Algorithmic Combinatorics | Wi 2016–17
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Exercise Sheet 14

Due date: 14:00, Feb 14th, by the end of the lecture.
Late submissions will be used to shield Earth from nasteroids.[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 This is some meaningless text, only present because I do not like jumping directly into an "itemize" environment.

(a) Let $\mathcal{F}$ be a hypergraph with the property that for any $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \geq 2$. Prove that $\mathcal{F}$ is two-colourable.

(b) Deduce that if G is a bipartite graph, and L is an assignment of lists of colours to the vertices with the property that for any two vertices $x, y \in V(G)$, $|L(x) \cap L(y)| \geq 2$, then G has a proper L-colouring.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S14.html`.]

Bonus (10 pts) If G is a bipartite graph with $|V(G)| = n \geq 3$, prove that $\chi_\ell(G) \leq \lceil \log_2 n \rceil$.

Exercise 2 The goal of this exercise is to prove $m(k) = O(k^2 2^k)$ via a randomised construction. Fix the ground set of elements $[2n]$, and consider the random hypergraph $\mathcal{F}$ with m edges, $F_1, F_2, \ldots, F_m$, chosen independently and uniformly at random (with repetition) from the family $\binom{[2n]}{k}$ of all k-sets in $[2n]$.

(a) Let χ be a (fixed) red/blue colouring of the elements $[2n]$. Show that, for each i, the probability p of F_i being monochromatic is at least $2\binom{n}{k}/\binom{2n}{k}$.

(b) Provided $x > 0$ is sufficiently small, $1 - x \leq e^{-x} \leq 1 - \frac{1}{2}x$. Using this, deduce that $p \geq 2^{-k} e^{-2k^2/(2n-k)}$ if n is sufficiently large with respect to k.

(c) Deduce that the probability that $\mathcal{F}$ is two-colourable is at most $2^{2n}(1-p)^m$.

(d) By choosing appropriate values for m and n in terms of k, show that there exists a non-two-colourable k-graph $\mathcal{F}$ with $O(k^2 2^k)$ sets.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S14.html`.]

[1] The general consensus is that asteroids that try to destroy our planet are not very nice. The community of people who worry about these asteroid impacts have since shortened the term "nasty asteroids."

Exercise 3 Show that our bound from the Erdős–Selfridge corollary is tight; that is, for all $k \geq 2$, $\tilde{m}(k) = 2^{k-1}$.

Exercise 4 In this exercise we consider the strong game played on the hypergraph $(X, \mathcal{F})$, as defined in lecture.

(a) Prove that either one player has a winning strategy,[2] or that both players have drawing strategies.[3]

(b) Prove that Second Player cannot have a winning strategy.

Exercise 5 In the strong (n,t)-Ramsey game, we have $X = E(K_n)$, and the winning sets are given by $\mathcal{F} = \{E(K) : K \text{ is a } t\text{-clique in } K_n\}$. We can think of this game as being played on the complete graph K_n; the players alternatively colour the edges of the complete graph, and the first player to create a monochromatic K_t wins the game.

(a) Prove that for every $t \geq 2$, there is some finite $n_0(t)$ such that whenever $n \geq n_0(t)$, First Player has a winning strategy for the strong (n,t)-Ramsey game.

(b) Show that $n_0(3) = 5$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S14.html`.]

Exercise 6 Suppose we modify the rules of the Maker–Breaker game on a k-uniform hypergraph $(X, \mathcal{F})$: while Maker still selects one element of X in every turn, Breaker can now select q elements in every turn. Their goals remain the same: Maker wins if she selects all k elements from some winning set $F \in \mathcal{F}$, while Breaker wins if he selects at least one element from every winning set $F \in \mathcal{F}$.

Show that whenever $|\mathcal{F}| < (q+1)^{k-1}$, Breaker has a winning strategy, and describe this winning strategy explicitly.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S14.html`.]

[2] That is, a strategy guaranteeing that he or she will win the game, regardless of what the opponent does.
[3] That is, both players can guarantee that they will not lose, regardless of their opponent's moves.