Algorithmic Combinatorics — Wi 2016–17

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Exercise Sheet 15

Due date: This practice sheet is not for submission.
(No due date ⇒ no late submissions ⇒ no dire warning.)

You may demonstrate your understanding and mastery of the topics from the last week of lectures — the Lovász Local Lemma and its algorithmic version — by solving the exercises below. Solutions will shortly be available on the course website, if they are not already there.

Exercise 1 Recall the statement of the Lovász Local Lemma we had in class.

> **Theorem 1** (Lovász Local Lemma)**.** *Let $E_1, E_2, \ldots, E_m$ be events in some probability space. Let $d \in \mathbb{N}$ and $p \in [0,1]$ be such that, for every $i \in [m]$, we have*
>
> *(1) $\mathbb{P}(E_i) \le p$, and*
>
> *(2) there is a set $\Gamma(i) \subseteq [m] \setminus \{i\}$ of at most d indices, such that the event E_i is mutually independent of $\{E_j : j \in [m] \setminus (\Gamma(i) \cup \{i\})\}$.*
>
> *If $ep(d+1) \le 1$, then with positive probability none of the events E_i occur.*

In this exercise you will prove Theorem 1.

(a) Show that for any $i \in [m]$ and $J \subseteq [m] \setminus \{i\}$, we have $\mathbb{P}\left(E_i | \cap_{j \in J} E_j^c\right) \le ep$. You may use the estimate $(1 - 1/(d+1))^d \ge e^{-1}$.

(b) Deduce that $\mathbb{P}\left(\cap_{i \in [m]} E_i^c\right) \ge (1 - ep)^m > 0$.

Exercise 2 Recall that the Ramsey number $R(k,k)$ is the smallest n such that any two-colouring of the edges of K_n must contain a monochromatic copy of K_k.

(a) By colouring edges randomly, show that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. Deduce that $R(k,k) \ge \frac{1}{e\sqrt{2}}(1 + o(1)) k 2^{k/2}$. [This is from Discrete Math I.]

(b) Obtain a $\sqrt{2}$-factor improvement of the result in (a) by 'correcting' a random colouring by removing a vertex from every monochromatic clique: show that for any integer n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. Deduce that $R(k,k) \ge \frac{1}{e}\left(1 + o(1)\right) k 2^{k/2}$.

(c) Improve the bound by yet another $\sqrt{2}$-factor with the Local Lemma: show that if $e\binom{k}{2}\binom{n-2}{k-2} 2^{1-\binom{k}{2}} \le 1$, then $R(k,k) > n$. Deduce the bound $R(k,k) \ge \frac{\sqrt{2}}{e}(1 + o(1)) k 2^{k/2}$.

Exercise 3 A *Boolean variable* can take one of two values — either *true* or *false*. Given a variable x, its negation $\neg x$ takes the opposite value. A *literal* is either a variable or its negation. A *k-clause* is the 'or' of k literals corresponding to distinct variables, and is true if and only if at least one of its literals evaluates to true. Finally, a *k-SAT formula* is the 'and' of a number of k-clauses, and is true if and only if all of its clauses are true. For example, for the 3-SAT formula

$$f(x_1, x_2, x_3, x_4) = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4),$$

we have $f(T, T, T, T) = T$ and $f(F, F, F, F) = T$, but $f(T, F, T, T) = F$, where T represents true and F represents false. In general, we say that a k-SAT formula is *satisfiable* if there is some input for which it evaluates to being true, and so our example f is satisfiable.

Prove that every k-SAT formula where no variable appears in more than $\frac{2^k}{ek}$ clauses is satisfiable.

Exercise 4 The city of London is surrounded by the M25 motorway, a circular road that directs traffic around the city without congesting its inner roads. It is approximately 110 miles long and, as per UK traffic regulations, has 30 streetlights per mile, and thus a total of 3300 lampposts.

To comply with recent environmental guidelines, the Mayor of London wants to illuminate the M25 with environmentally-friendly lightbulbs that will consume less power while maintaining adequate light coverage. To find the best lightbulb for the job, he commissions London's 300 different lighting firms to submit prototypes for evaluation.

Each firm provides a sample of 11 lightbulbs. To ensure that no firm has all its lightbulbs in a favourable stretch of the highway, all 3300 lightbulbs are mixed together and then placed, in some arbitrary order, in the M25's lampposts. The Mayor intends to keep these lightbulbs in place for a month and evaluate their efficiency before making a final decision about which lightbulb to use in the long-term.

Unfortunately, after a few days, he realises that this experiment is rather expensive, and decides the test has to be scaled down[1]. Thus one of each company's 11 lightbulbs will be switched off. However, in the interests of public safety, no two neighbouring lightbulbs should both be switched off, for fear of creating too long a dark stretch on the motorway.

Show that, regardless of how the lightbulbs were initially distributed, it is always possible to safely turn off one lightbulb from each company.

Exercise 5 In the notes on the algorithmic Local Lemma, it is shown that for a k-uniform hypergraph $\mathcal{F}$ with n vertices and $\Delta(L(\mathcal{F})) \leq 2^{k-4}$, the expected number of recolourings in the algorithmic Local Lemma is $O(m \log m)$. With more careful analysis, show that this bound can be greatly improved to $O\left(\frac{n}{k} \log m\right)$.

[1] An alternative would have been to raise taxes to fund the project, but he is a proud patriot, and, after a rather poor showing at the UEFA Euro 2016, decides his country can ill afford to surrender either of her two advantages over France: lower taxes and finer food.