Algorithmic Combinatorics — Wi 2016–17

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Exercise Sheet 3

Due date: 14:00, Nov 8th, by the end of the lecture.
Late submissions will be mailed in as votes for Trump.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Just as we extended the problem of finding a spanning tree to weighted graphs, so too can we ask for maximum weight matchings in weighted graphs. In this setting we are given a complete graph $K_{n,n}$, with vertex classes $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, together with nonnegative edge weights $\omega_{i,j} = \omega(\{x_i, y_j\}) \geq 0$ for all $1 \leq i, j \leq n$. The weight of a matching M is given by $\omega(M) = \sum_{e \in M} \omega(e)$, and the problem is now to find a perfect matching of maximum weight.

(a) Show that the problem of finding a maximum matching in a bipartite graph can be reduced to finding a maximum-weight matching in a complete bipartite graph.

We can also have price functions for the vertices, with $u(x_i) = u_i$ and $v(y_j) = v_j$. A pair of price functions (u, v) is called a *weighted cover* if $u_i + v_j \geq \omega_{i,j}$ for all $1 \leq i, j \leq n$. The *cost* of the cover (u, v) is given by $c(u, v) = \sum_{i=1}^n u_i + \sum_{j=1}^n v_j$.

(b) Prove that for every perfect matching M and every weighted cover (u, v), we have $\omega(M) \leq c(u, v)$. Moreover, we have equality if and only if there is some permutation $\pi \in S_n$ such that $M = \{\{x_i, y_{\pi(i)}\} : i \in [n]\}$ and $u_i + v_{\pi(i)} = \omega_{i,\pi(i)}$ for all i.

Exercise 2 In this exercise, you will see that both Hall's and Tutte's theorems can be extended to give certificates for maximum matchings and not just perfect matchings.

(a) Prove that for every bipartite graph $G = (X \cup Y, E)$, we have

$$\alpha'(G) = \min_{S \subseteq X} \left(|X| - |S| + |N(S)| \right).$$

(b) Prove that for every graph $G = (V, E)$, we have

$$2\alpha'(G) = \min_{S \subseteq V} \left(|V| + |S| - o(G \setminus S) \right).$$

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S03.html`.]

Exercise 3 Give an inductive proof of Hall's theorem — that a bipartite graph $G = (X \cup Y, E)$ has a matching saturating X if and only if for every subset $S \subseteq X$ we have $|N(S)| \geq |S|$ — that makes no mention of augmenting paths.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S03.html`.]

Exercise 4 A graph $G = (V, E)$ is called *cubic* if every vertex has degree three, and *bridgeless* if we have to remove at least two edges to disconnect G.

(a) Prove that every cubic bridgeless graph has a perfect matching.

(b) Give an example of a cubic graph that does not have a perfect matching.

Exercise 5 Given a graph $G = (V, E)$, one could try to apply the greedy algorithm to find a maximum matching of G. Order the edges $E = \{e_1, e_2, \ldots, e_m\}$ in some (arbitrary) way, and start with $M_0 = \emptyset$. At time t, for every $1 \leq t \leq m$, if $M_{t-1} \cup \{e_t\}$ is a matching, set $M_t = M_{t-1} \cup \{e_{t-1}\}$, and otherwise set $M_t = M_{t-1}$. Return the final matching M_m.

Prove that this gives a $\frac{1}{2}$-approximation algorithm for the maximum matching problem, and give an example to show that the $\frac{1}{2}$ approximation ratio is tight for this algorithm.

Exercise 6 A *matroid* is a pair $(X, \mathcal{B})$ of a finite ground set X and a collection $\mathcal{B}$ of subsets of X called *bases* that satisfy the following axioms:

(A1) There is at least one basis, i.e. $\mathcal{B} \neq \emptyset$.

(A2) The basis exchange property: If $A, B \in \mathcal{B}$ with $A \neq B$, then for every $a \in A \setminus B$ there is some $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ is another basis.

A set A is called *independent* if it is a subset of some basis; that is, if there is $B \in \mathcal{B}$ with $A \subseteq B$. You are already familiar with some fundamental matroids: for example, the set of bases of a finite vector space, or the set of spanning trees in a connected graph. Start by showing the following is true in this more general setting.

(a) Show that all bases in a matroid must have the same cardinality.

If we assign nonnegative weights $w : X \to \mathbb{R}_{\geq 0}$ to elements in the ground set, we can then ask for the minimum weight basis; that is, for $B \in \mathcal{B}$ minimising $\sum_{x \in B} w(x)$.

The greedy algorithm is as follows: let X be ordered by weight, so $X = \{x_1, x_2, \ldots, x_n\}$ with $w(x_1) \leq w(x_2) \leq \ldots \leq w(x_n)$. Start with $S_0 = \emptyset$. At time t, for $1 \leq t \leq n$, let $T = S_{t-1} \cup \{x_t\}$. If T is independent, set $S_t = T$, and otherwise set $S_t = S_{t-1}$. The output of the algorithm is the final set S_n.

(b) Prove that the greedy algorithm produces a basis of minimum weight.

Bonus (10 pts) Prove that the converse of (b) is also true: if $\mathcal{F}$ is a collection of subsets of a finite ground set X such that for any nonnegative weight function $w : X \to \mathbb{R}_{\geq 0}$, the greedy algorithm always produces a set $F \in \mathcal{F}$ of minimal weight, then $(X, \mathcal{F})$ is a matroid.