Algorithmic Combinatorics — Wi 2016–17
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Exercise Sheet 4

Due date: 14:00, Nov 15th, by the end of the lecture.
Late submissions will be ... what could be worse than this week's events?[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Use the Hungarian Algorithm, showing all the key steps, to find a *minimum*-weight matching in $K_{5,5}$ with the edge weights $W = (\omega_{i,j})$ as below, and then give a short proof that your matching is optimal.

$$W = \begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}.$$[2]

[1] That said, late homework will still not be accepted, so don't be late, people.

[2] "Come now," you complain, "what is the meaning of this? Am I some mindless machine, with nothing better to do than to thoughtlessly repeat the steps of some algorithm? Thanks but no thanks — I have better things to do," you conclude dramatically as you start the next level of Candy Crush.

While we question your choice of pastime, we concede that you have a point. Simply calculating a minimum-weight matching in some arbitrary small weighted bipartite graph does not make for the most interesting of exercises, even though the act of actually working through the algorithm could lead you to better understand it. Still, this is no ordinary weighted graph, but one with grave import, as we shall now explain.

You see, this exercise concerns Luise, whom we join as she walks aimlessly through a forest. Under normal circumstances she would delight in the rich variety of autumnal colours to be beheld both overhead and underfoot, but on this occasion her mind was elsewhere, weighed down by other matters.

She was slightly startled, then, when a voice rang out from somewhere behind her. "Why, you're not wearing any shoes!"

Indeed she was not, for she enjoyed walking through mud barefoot, but she did not feel the need to explain her choice of footwear (or lack thereof) to a stranger and, still deep in her own thoughts, did not turn around to address this rather direct stranger.

However, the voice was not to be discouraged by her obvious disinterest in conversation. "Well, it is not very common for me to see a human without shoes, but it's nice that we have this in common. It is a pleasure to meet you. My name is Happy Bubu, and I am the happiest cat in Albania! I might be the happiest cat in the world," he laughed, "but I have not been outside Albania, so it is difficult for me to make that claim."

Now Luise's curiosity was piqued, for she was certain she must have misheard something. She turned around and faced the cat, whom she found to be sporting a wide and cheerful grin. "Did I just hear you say

that you have never left Albania?"

"Indeed," came Happy Bubu's response, "while I would like to one day see more of the world, I have not yet had the opportunity. I grew up in Tirana, where I stayed with my wonderful owner. She took brilliant care of me, so it is no wonder that I am always so happy. However, a cat like me belongs in wide open spaces like this," he explained, waving his tail at the forest they were surrounded by, "so I eventually moved to a farm in the countryside. I miss her very much, of course, but life here is good. For example, the mice here run freely around on the ground, and are much more fun to chase than the ones in Tirana, which would just sit next to my owner's computer."

As Happy Bubu stopped to draw breath, Luise took the long-awaited opportunity to step in. "How can you say you have never left Albania when we're in Berlin?"

It was Happy Bubu's turn to be confused. "You must be mistaken, my dear ..." his voice trailed off as he realised he did not know Luise's name.

"Luise," said Luise.

"... Luise," continued Happy Bubu, "for we are most certainly in Albania. If we were to walk just half an hour in that direction," he pointed north with a whisker, "we would reach the farmhouse I now call home."

Luise quickly pulled out her phone, and, seeing that she was now roaming with Eagle Mobile, realised that Happy Bubu must be speaking the truth. "Oh no," she exclaimed, "I wanted to take a walk through the Grunewald to sort some things out, but it's been more difficult than I'd thought, and I must have ended up here in the meanwhile."[3]

"You must have some pretty serious problems," mused Happy Bubu, "if they are capable of distracting you for so long. Tell me what troubles you so. Happy Bubu will help."

Luise did not normally confide in cats she had only just met, but since she had apparently walked through Europe without realising it, she reckoned she could use all the help she could get. "Well, I was playing Looping Louie with my friend the other day. I always win, and did not want to upset her, so I thought I would let her win one game."

"That sounds very nice of you," said Happy Bubu, who was always quick to appreciate niceness in others. "What could possibly be the problem? You shouldn't be so upset about losing a single game."

"It's not that," explained Luise, "but we make little bets about our games. After I lost, I was expecting she would ask me to bake her a cake, which I would have been happy to do. Instead, she's forcing me to go out dancing with her."

"That sounds even better," said Happy Bubu, positively beaming, "everyone loves a party! What kind of music do you like? Say, have you heard this new album, 'Songs of the Ood'?"

"I don't have time for such things," cried Luise, ignoring his questions, "I have to attend lectures, and do my homework, and go to work, and meet my other friends. Oh, and our algorithmic homework assignment will probably have a ton of footnotes that will take hours to read, so that counts as its own thing."

For once, Happy Bubu's smile faltered. "Goodness," he gasped, "that sounds like quite the burden. How do you ever get through all of that? Do you have a time turner?" Fearing that she might not get the dated reference, he added, "You know, like what Hermione used to get to all of her classes in the Harry Potter series."

"Haha," Luise laughed, listlessly, "no. I guess I am just pretty efficient at getting things done. However, no matter how I try to arrange things, it seems impossible to squeeze in a night of dancing amid all my other tasks. Any suggestions?"

Fortunately, Happy Bubu was never short of ideas. "I know," he shouted, excitedly, "you could hire help! Sometimes when there are too many mice for me to catch by myself, I call the cat that lives next door, and we chase them all together. Although I imagine that in your case you'll need more than one person to help out with your superhuman workload. In fact, you should probably hire five different people, with one task per person, so as to avoid overwhelming them. Now of course, different people have different skills, so they would charge you different rates for different tasks — one might be willing to take notes in your lectures

Exercise 2 In class we showed that the Hungarian Algorithm terminates when the weights are rational, but we did not give an effective upper bound on the number of iterations required. Prove that the Hungarian Algorithm requires at most n^2 iterations, even with weights in $\mathbb{R}_{\geq 0}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]

Exercise 3 Your intuition may suggest that the Gale–Shapley Proposal Algorithm is best for the men, since they choose whom to propose to, and worst for the women, who can only respond to the offers they get.[4]

(a) Formulate this intuition as a mathematically precise statement.

(b) Prove that your statement in (a) is correct.

Bonus (5 pts) How bad can things get for the men? Determine the maximum $k = k(n)$ such that there is a set of preference lists for which the Proposal Algorithm does not match any man with one of his top k partners.

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at a cheap price, but charge you a lot for hanging out with your friends, but another person might be the complete opposite."

"I say," said Luise, "I never considered that possibility. I'd normally like to take care of everything by myself, but that seems truly impossible, so I'll look into this." So saying, she called five different agencies, and got their quotes for each task, resulting in the matrix of this problem. However, one hurdle remained. "How should I assign the tasks to the agencies? I don't want this to get too expensive."

Just as Happy Bubu was about to explain the intricacies of the Hungarian Algorithm, they heard someone crying in the distance. "I would love to help you further, Luise, but it sounds like there is someone who needs my happiness more. I wish you the best of luck. Lamtumirë." And in a flash Happy Bubu disappeared in the direction of the sobs, leaving nothing but the melodic sound of his joyous laughter in his wake.

Left alone once again, Luise turns to you for help. Can you find a way to cheaply assign her five chores to the five agencies, enabling her to go dancing with her friend?

[3]Happy Bubu was going to point out the physical impossibility of such a walk, but then remembered that he was a talking cat.

[4]That an arbitrary choice of which people should do the proposing should have such dire consequences for members of the opposite gender is, no doubt, deeply unfair, and has inspired several activists to encourage women to improve their situation by taking a more active role in the process. As with Al Gore's environmental efforts, though, musical methods have led to the greatest success. The primary example is perhaps Beyoncé's hit song 'Single Ladies', which has the encouraging lyrics, "if you liked it then you should've put a ring on it."

The fact that the next line in the song starts "wuh uh oh," though, hints at a new problem that arises. Imagine, if you will, that we adapt the algorithm to allow both men and women to propose to their top remaining choice at every step. Every evening, then, the people of this town will buy bouquets of roses and boxes of chocolate, stash diamond rings in their pockets, heft boomboxes up on their shoulders and then drive to their dearly beloved's residence. Except, of course, when they arrive there will be no one at home, for the intended target of their affections will be off trying to woo their own dream partner, and so all this effort will be for naught.

Exercise 4 If you forget the Proposal Algorithm, you might try to find a stable matching by using the Hungarian Algorithm instead. Given men $\{M_1, M_2, \ldots, M_n\}$ and women $\{W_1, W_2, \ldots, W_n\}$, each with their own preference lists of the members of the opposite gender, a natural edge-weighting would be $\omega(\{M_i, W_j\}) = 2n - k - \ell$, where M_i is the kth man on W_j's list, and W_j is the ℓth woman on M_i's list.

Show that for every (large enough, if needed) n, there are preference lists such that no maximum-weight matching is a stable matching.

Exercise 5 Suppose we wish to find stable matchings between n men and n women.

(a) Construct preference lists for which there are at least $2^{\lfloor n/2 \rfloor}$ stable matchings.

(b) Improve this for $n = 2^k$ by giving preference lists with at least 2^{n-1} stable matchings.

(c) Use this to show that for any n we can have at least $\Omega\left(2^n/n\right)$ stable matchings.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]

Exercise 6

(a) Define degen(G) to be the minimum d such that in every subgraph $H \subseteq G$, there is a vertex $v \in V(H)$ with at most d neighbours in H. Prove that for an appropriate ordering of the vertices, the Greedy Algorithm uses at most degen(H) + 1 colours to properly colour G.

(b) Show that for any graph with m edges, $\chi(G) \leq \frac{1}{2}\left(\sqrt{8m+1}+1\right)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]