Algorithmic Combinatorics Wi 2016–17
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Exercise Sheet 4

Due date: 14:00, Nov 15th, by the end of the lecture.
Late submissions will be ... what could be worse than this week's events?[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Use the Hungarian Algorithm, showing all the key steps, to find a *minimum*-weight matching in $K_{5,5}$ with the edge weights $W = (\omega_{i,j})$ as below, and then give a short proof that your matching is optimal.

$$W = \begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}.$$

Exercise 2 In class we showed that the Hungarian Algorithm terminates when the weights are rational, but we did not give an effective upper bound on the number of iterations required. Prove that the Hungarian Algorithm requires at most n^2 iterations, even with weights in $\mathbb{R}_{\geq 0}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]

Exercise 3 Your intuition may suggest that the Gale–Shapley Proposal Algorithm is best for the men, since they choose whom to propose to, and worst for the women, who can only respond to the offers they get.

(a) Formulate this intuition as a mathematically precise statement.

(b) Prove that your statement in (a) is correct.

Bonus (5 pts) How bad can things get for the men? Determine the maximum $k = k(n)$ such that there is a set of preference lists for which the Proposal Algorithm does not match any man with one of his top k partners.

[1] That said, late homework will still not be accepted, so don't be late, people.

Exercise 4 If you forget the Proposal Algorithm, you might try to find a stable matching by using the Hungarian Algorithm instead. Given men $\{M_1, M_2, \ldots, M_n\}$ and women $\{W_1, W_2, \ldots, W_n\}$, each with their own preference lists of the members of the opposite gender, a natural edge-weighting would be $\omega(\{M_i, W_j\}) = 2n - k - \ell$, where M_i is the kth man on W_j's list, and W_j is the ℓth woman on M_i's list.

Show that for every (large enough, if needed) n, there are preference lists such that no maximum-weight matching is a stable matching.

Exercise 5 Suppose we wish to find stable matchings between n men and n women.

(a) Construct preference lists for which there are at least $2^{\lfloor n/2 \rfloor}$ stable matchings.

(b) Improve this for $n = 2^k$ by giving preference lists with at least 2^{n-1} stable matchings.

(c) Use this to show that for any n we can have at least $\Omega\left(2^n/n\right)$ stable matchings.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]

Exercise 6

(a) Define $\text{degen}(G)$ to be the minimum d such that in every subgraph $H \subseteq G$, there is a vertex $v \in V(H)$ with at most d neighbours in H. Prove that for an appropriate ordering of the vertices, the Greedy Algorithm uses at most $\text{degen}(H) + 1$ colours to properly colour G.

(b) Show that for any graph with m edges, $\chi(G) \leq \frac{1}{2}\left(\sqrt{8m+1} + 1\right)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S04.html`.]