Algorithmic Combinatorics — Wi 2016–17

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Exercise Sheet 5

Due date: 14:00, Nov 22nd, by the end of the lecture.
Late submissions will most mysteriously disappear.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In lecture we saw that $\chi_\ell(K_{2,2}) = 2$. Generalise this result by proving that the complete k-partite graph $K_{\underbrace{2,2,\ldots,2}_{k \text{ parts}}}$ with k parts each of size two has $\chi_\ell(K_{2,2,\ldots,2}) = k$.

Bonus (5 pts) It is a well-known fact[1] that two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices have $\chi(G_1 \cup G_2) \le \chi(G_1)\chi(G_2)$, where $G_1 \cup G_2 = (V, E_1 \cup E_2)$. Show that this inequality extends to the list-chromatic setting: $\chi_\ell(G_1 \cup G_2) \le \chi_\ell(G_1)\chi_\ell(G_2)$.

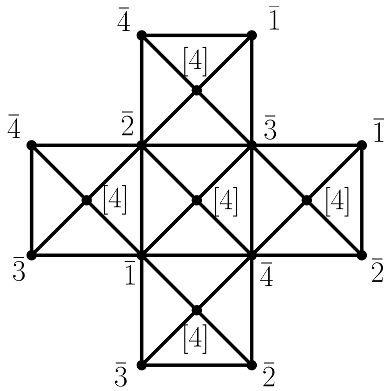
Exercise 2 Prove that any bipartite graph G with non-empty lists $L = \{L(v) : v \in V(G)\}$ satisfying $|L(u) \cap L(v)| \ne 1$ for every edge $\{u, v\} \in E(G)$ has a proper L-colouring.

Exercise 3 Show that every triangle-free planar graph is 4-choosable.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S05.html`.]

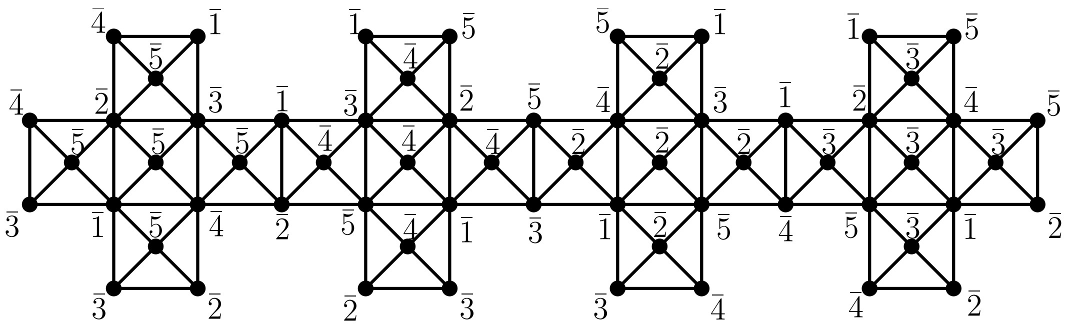
Exercise 4 In this exercise we will prove that Thomassen's theorem is tight; that is, there are planar graphs that are not 4-choosable.

(a) Show that the following graph has no proper coloring from the assigned lists. (In this part of the exercise, $\bar{i}$ denotes the set $[4] \setminus \{i\}$.)



[1] If you have not seen this before, you should enjoy proving it.

(b) Let G be the graph obtained from the graph below by adding an extra vertex to the outside face and connecting it to all the vertices on the boundary. Show that if the new vertex is assigned the list $\bar{1}$, then there is no proper list coloring of G from the assigned lists. (In this part of the exercise, $\bar{\imath}$ denotes the set $[5] \setminus \{i\}$.)



Exercise 5

(a) Prove that for every bipartite graph G we have $\chi'(G) = \Delta(G)$.

(b) Let $\vec{D} = (V, \vec{E})$ be a directed graph where every vertex has at most d in-edges and at most d out-edges.[2] Show that the edges of $\vec{D}$ can be d-coloured in such a way that edges with either the same start-vertex or the same end-vertex receive different colours.

Exercise 6 Let G be a connected graph with an even number of edges. Prove that the edges of G can be decomposed into an edge-disjoint union of copies of $P_{2.5}$.[3]

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S05.html`.]

[2] For a vertex v, an in-edge is an edge of the form (u, v), while an out-edge is an edge of the form (v, u).
[3] Following modern conventions, $P_{m+0.5}$ is a path with m edges (and thus $m + 1$ vertices).