Algorithmic Combinatorics — Wi 2016–17
Shagnik Das — Tibor Szabó

Exercise Sheet 6

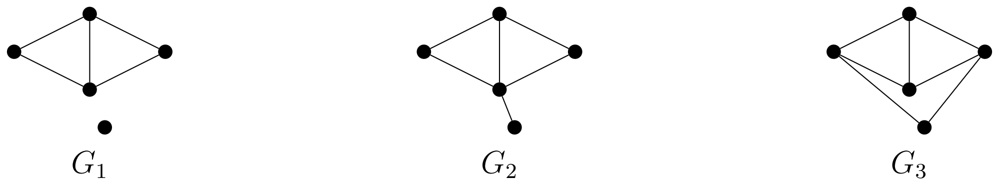
Due date: 14:00, Nov 29th, by the end of the lecture.
Late submissions will be stuffed into a turkey.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 A *clique decomposition* of a multigraph G is a collection $\mathcal{K}$ of cliques[1] in G such that every edge of G is contained in at least one clique in $\mathcal{K}$.

(a) Prove that G is the line graph of some multigraph H if and only if G has a clique decomposition $\mathcal{K}$ with every vertex appearing in exactly two cliques in $\mathcal{K}$. Show further that H may be taken to be a simple graph if and only if every pair of vertices in G appear together in at most one clique of $\mathcal{K}$.

(b) For the three graphs G_i below, provide, if possible, a multigraph H_i for which we have $G_i = L(H_i)$. If possible, make H_i a simple graph. Whenever these tasks are impossible, explain why.



Exercise 2 Use Petersen's Theorem on 2-factors in regular graphs of even degree to prove that every multigraph G satisfies $\chi'(G) \leq 3\lceil \Delta(G)/2 \rceil$, and show this bound can be tight for even values of $\Delta(G)$.

Exercise 3 Let G be a graph with maximum degree $\Delta = \Delta(G)$.

(a) Show that if the vertices in G of degree Δ induce a forest, then G is Δ-edge-colourable.

(b) Show that for every matching M in G, there is a $(\Delta + 1)$-edge-colouring of G in which all edges of M receive the same colour.

[1] A clique is a complete subgraph, and we allow cliques with just a single vertex.

Exercise 4 Let G be a bipartite graph. An *orientation* of G is a directed graph $\vec{D}$ where every edge $\{u,v\}$ of G is assigned *one* of the orientations (u,v) or (v,u) in $\vec{D}$.

(a) Let $\vec{D}$ be an orientation of G where every vertex has an out-neighbour. Show that $\vec{D}$ has a kernel.

(b) Show that every orientation $\vec{D}$ of G has a kernel.

(c) Deduce that every bipartite digraph is kernel-perfect.

Bonus (5 pts) Show that every bipartite graph with maximum degree Δ is $\left(\lceil\frac{\Delta}{2}\rceil+1\right)$-list-colourable.

Exercise 5 Your mission, should you choose to accept it, is to extend the proof of Galvin's theorem from $K_{n,n}$ to general bipartite graphs.

(a) Considering the complete bipartite graph $G = K_{n,n}$ with vertices $X \cup Y$, show that the edge-colouring $\phi(\{x_i, y_j\}) = i + j \pmod n$ gives a proper n-edge-colouring of G.

(b) Let G now be an arbitrary bipartite graph with edge-chromatic number $\chi'(G) = n$. Use an n-colouring of $E(G)$ to find an orientation $\vec{D}$ of the line graph $L(G)$ such that:

  (i) The maximum out-degree is at most $n-1$.

  (ii) Given an edge $\{x,y\}$ of G, the digraphs induced on the sets $S_x = \{\{x,y'\} : y' \in N(x)\} \subset V(L(G))$ and $S_y = \{\{x',y\} : x' \in N(y)\} \subset V(L(G))$ are transitive.

(c) Show that $\vec{D}$ is kernel-perfect, and deduce that $\chi'_\ell(G) = n$.

Exercise 6 We know that distinguishing between Class 1 ($\chi'(G) = \Delta(G)$) and Class 2 ($\chi'(G) = \Delta(G)+1$) graphs is generally hard to do. In this exercise you will provide a (very) partial classification.

(a) Show that if G is a regular graph with a cut-vertex, then G is a Class 2 graph.

For the following exercises, we shall study the Cartesian product of graphs, $G\square H$. This product is defined on the vertices $V(G\square H) = V(G) \times V(H)$, with edges $(g_1,h_1) \sim (g_2,h_2)$ if and only if either $g_1 = g_2$ and $h_1 \sim_H h_2$, or $g_1 \sim_G g_2$ and $h_1 = h_2$.[2]

(b) Prove that for any graph H, $K_2\square H$ is a Class 1 graph.

(c) Extend part (b) by showing that whenever G is a Class 1 graph, $G\square H$ is as well.

[2] This product is denoted with the symbol $\square$ to denote the fact that $K_2\square K_2 = C_4$.