Algorithmic Combinatorics — Wi 2016–17
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Exercise Sheet 8

Due date: 14:00, Dec 13th, by the end of the lecture.
Late submissions will be turned into papier-mâché and moulded into casts.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 The following table describes a network.

Start	→	End	Capacity (€, mil)	Start	→	End	Capacity (€, mil)
Mad	→	Ams	50	Mad	→	Dub	140
Ams	→	Jer	40	Ams	→	Lux	40
Ams	→	Ber	50	Jer	→	Lux	30
Dub	→	Ams	130	Dub	→	Zür	20
Dub	→	Cay	50	Ber	→	Dub	60
Ber	→	Zür	60	Lux	→	Ber	40
Lux	→	Zür	30	Lux	→	BVI	60
Cay	→	Zür	30	Zür	→	BVI	120

Find[1] a flow of maximum possible value from Madrid ('Mad') to the British Virgin Islands

[1]You might wonder why you are being asked to do this, for it does not appear to be a particularly inspiring task, and you are not learning higher mathematics to simply run through some algorithms like a soulless machine.[2] You might also wonder why the capacities of the edges have as units millions of Euros. Does this exercise arise from some important real-world application?

Indeed it does. In this exercise, you are playing the role of a tax adviser to the great Cristiano Ronaldo.[3] Your client has a salary of approximately €950000 per week,[10] 52% of which is dutifully paid in tax to the Spanish government. Thus Ronaldo has about €23600000 to take home every year, a measly amount on which no self-respecting football star can be expected to live.

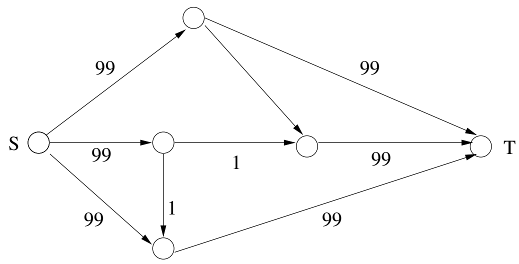
Fortunately for your client, he has a chiselled jaw, malleable hair, abs that could have been carved by Michaelangelo himself, and a rather large international fan club, and thus makes a lot of money from advertisements and other entrepreneurial endeavours. Seeing as how this money is earned internationally, and not just within Spanish borders, you explain to him that there is no reason he should put 52% of it into the Spanish Treasury, and suggest that he instead divert it towards the British Virgin Islands, which has rather more forgiving views on taxes.

Cristiano looks appalled at your suggestion that he should evade his taxes,[11] but is appeased when you explain that there is nothing nefarious about this, and that the more he saves on taxes, the more he can spend on his altruism. Your argument clearly strikes a chord with the man who was last year named the world's most charitable sportsperson,[12] and he agrees to hear you out.

You explain that it is preferable to move the money between some number of friendly financial hubs, and pull up the table above, showing the maximum amount that can be safely transferred between accounts. You are about to launch into a long, technical description of the financial mechanisms involved when he interrupts: "Fine, fine, but how much money in total can I save for charity?"

('BVI'), and give a short proof that one cannot do better.

Exercise 2 Consider the network in the figure below. The source and the sink are marked with S and T, and the capacities of all but one edge are indicated. The remaining edge has capacity $\frac{1}{2}(\sqrt{5}-1)$.



(a) Find (with proof) the value of the maximum flow in the network.

(b) Describe a choice of augmenting paths in the Ford-Fulkerson algorithm for which the algorithm never finishes and the flow value converges to $2+\sqrt{5}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S08.html`.]

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[2]Although perhaps you ought not to be so picky, because one day soon all tasks will be performed by soulless machines, for which we humans will merely be a source of energy, until something better comes along and we can be dispatched of entirely.

[3]Since it appears that you are not all football fans, perhaps I ought to explain that Cristiano Ronaldo is a Portuguese footballer who plays for Real[4] Madrid, which is a football club based in Madrid. Ronaldo[5] is excellent at what he does, and is widely considered to be one of the top two[9] footballers in today's game, and perhaps even in history.

[4]The 'Real' in Real Madrid does not allude to the fact that this club is not imaginary, but is Spanish for 'Royal'.

[5]By which we mean Cristiano Ronaldo, and not Ronaldo Luís Nazário de Lima (usually only called Ronaldo), who was a very successful Brazilian who earlier also played for Real Madrid. At the time, the latter[6] Ronaldo was considered one of the great footballers of his generation, but the remarkable feats of the former[6] Ronaldo have made him the more prominent[7] Ronaldo, and so the latter is now often referred to as the Brazilian Ronaldo or Fat Ronaldo. However, those who don't like Cristiano will usually refer to the Brazilian as the Original Ronaldo or the Real[8] Ronaldo.

[6]Our use of 'former' and 'latter' refers to the order in which these footballers have been mentioned in our text, and not the order in which they appeared in world football.

[7]This can be observed by Googling 'Ronaldo' and getting results for Cristiano.

[8]This 'Real' is in English, although the pun works well.

[9]The other footballer is Lionel Messi, an Argentinean who plies his trade at FC Barcelona, the archrivals of Real Madrid. Messi fans generally like to vociferously denounce Ronaldo as the far inferior player, despite the fact that Messi has failed to score in the last six Clásicos (the fiercely contested matches between Real Madrid and Barcelona).

[10]No, that is not a typographical error. This is indeed a weekly salary.

[11]Avoid, you quickly correct, not evade. Nothing illegal going on here.

[12]See, for instance, `http://tinyurl.com/ronaltruism1` or `http://tinyurl.com/ronaltruism2`.

Exercise 3 Let G be a graph, and let $x, y \in V(G)$ be two vertices. By constructing an appropriate network $(\vec{D}, s, t, c)$, use the Ford-Fulkerson Theorem to prove $\kappa'(x, y) = \lambda'(x, y)$, without referring to any other versions of Menger's Theorem.

Exercise 4 In lecture we showed how one can prove the local, vertex version of Menger's Theorem; that is, given a graph G and non-adjacent vertices $x, y \in V(G)$, $\kappa(x, y) = \lambda(x, y)$. We constructed a directed graph $\vec{D}$ with vertices $V(\vec{D}) = \{v^+, v^- : v \in V(G)\}$ and edges $\vec{E}(\vec{D}) = \{(u^+, v^-), (v^+, u^-) : uv \in E(G)\} \cup \{(v^-, v^+) : v \in V(G)\}$. We then assigned edge capacities by setting

$$c(\vec{e}) = \begin{cases} 1 & \text{if } \vec{e} = (v^-, v^+), v \in V(G) \\ \infty & \text{otherwise} \end{cases},$$

and considered flows in the network $(\vec{D}, x^+, y^-, c)$.

To complete the proof, prove the following claim: the minimum capacity of an x^+, y^- cut in $\vec{D}$ is equal to the minimum size of an x, y-separating set in G.

Exercise 5 In this exercise you have the opportunity to perform the calculations needed in the inductive step in our proof of Baranyai's Theorem. Recall that for $1 \leq \ell \leq n$, we sought a collection of $M = \binom{n-1}{k-1}$ m-partitions $\mathcal{A}_i$ of $[\ell]$, where $m = \frac{n}{k}$, such that every set $F \subseteq [\ell]$ appeared (with multiplicity) in exactly $\binom{n-\ell}{k-|F|}$ of the m-partitions.

Given such a collection of m-partitions for $\ell \leq n-1$, we built a network $(\vec{D}, s, t, c)$, where $V(\vec{D}) = \{s, t\} \cup \{\mathcal{A}_i : i \in [M]\} \cup \{F : F \subseteq [\ell]\}$ and

$$\vec{E}(\vec{D}) = \{(s, \mathcal{A}_i) : i \in [M]\} \cup \{(\mathcal{A}_i, F) : i \in [M], F \in \mathcal{A}_i\} \cup \{(F, t) : F \subseteq [\ell]\}.$$

The capacities were given by

$$c(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F, t) \\ \infty & \textit{otherwise} \end{cases}.$$

(a) Prove that the flow f defined by

$$f(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \frac{k-|F|}{n-\ell} & \vec{e} = (\mathcal{A}_i, F) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F, t) \end{cases}$$

is indeed a feasible flow.

(b) We used an integral maximum flow to find a unique set $F_i \in \mathcal{A}_i$ for each $i \in [M]$, and then formed m-partitions $\mathcal{A}_i'$ of $[\ell+1]$ by adding the element $\ell+1$ to the set F_i in each $\mathcal{A}_i$. Show that this collection of m-partitions of $[\ell+1]$ satisfies the required conditions.

Exercise 6 Suppose $n \geq 2$. Baranyai's Theorem guarantees $\binom{[3n]}{3}$ can be partitioned into perfect matchings without explicitly describing these matchings. In this exercise you will give such an explicit description in the case when $p = 3n - 1$ is a prime number.

(a) Consider the field $\mathbb{F}_p$, and denote by $\mathbb{F}_p^*$ the set of invertible elements, namely $\mathbb{F}_p^* = \{1, 2, \ldots, p-1\}$. Define the map $\pi : \mathbb{F}_p^* \to \mathbb{F}_p$ by $\pi(x) = -(1+x)x^{-1}$. Show that π is injective and $\pi^3(x) = x$ for any $x \neq p - 1$.

(b) Add a new element u to $\mathbb{F}_p$, and extend π to $\{u, 0\}$ injectively so that $\pi^3(x) = x$ for all $x \in \mathbb{F}_p \cup \{u\}$. Show that this gives some perfect matching M_0 in $\binom{[3n]}{3}$.

(c) By considering affine transformations $x \mapsto ax + b$, find another $\binom{3n-1}{2} - 1$ perfect matchings in $\binom{[3n]}{3}$.

(d) Show that these matchings partition $\binom{[3n]}{3}$ into perfect matchings.