Exercise Sheet 8

Due date: 14:00, Dec 13th, by the end of the lecture.
Late submissions will be turned into papier-mâché and moulded into casts.

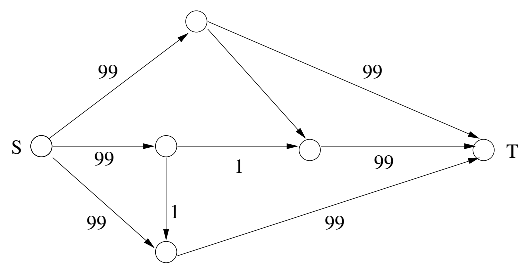
You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 The following table describes a network.

Start	→	End	Capacity (€, mil)	Start	→	End	Capacity (€, mil)
Mad	→	Ams	50	Mad	→	Dub	140
Ams	→	Jer	40	Ams	→	Lux	40
Ams	→	Ber	50	Jer	→	Lux	30
Dub	→	Ams	130	Dub	→	Zür	20
Dub	→	Cay	50	Ber	→	Dub	60
Ber	→	Zür	60	Lux	→	Ber	40
Lux	→	Zür	30	Lux	→	BVI	60
Cay	→	Zür	30	Zür	→	BVI	120

Find a flow of maximum possible value from Madrid ('Mad') to the British Virgin Islands ('BVI'), and give a short proof that one cannot do better.

Exercise 2 Consider the network in the figure below. The source and the sink are marked with S and T, and the capacities of all but one edge are indicated. The remaining edge has capacity $\frac{1}{2}(\sqrt{5}-1)$.



(a) Find (with proof) the value of the maximum flow in the network.

(b) Describe a choice of augmenting paths in the Ford-Fulkerson algorithm for which the algorithm never finishes and the flow value converges to $2+\sqrt{5}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S08.html`.]

Exercise 3 Let G be a graph, and let $x, y \in V(G)$ be two vertices. By constructing an appropriate network $(\vec{D}, s, t, c)$, use the Ford-Fulkerson Theorem to prove $\kappa'(x,y) = \lambda'(x,y)$, without referring to any other versions of Menger's Theorem.

Exercise 4 In lecture we showed how one can prove the local, vertex version of Menger's Theorem; that is, given a graph G and non-adjacent vertices $x, y \in V(G)$, $\kappa(x,y) = \lambda(x,y)$. We constructed a directed graph $\vec{D}$ with vertices $V(\vec{D}) = \{v^+, v^- : v \in V(G)\}$ and edges $\vec{E}(\vec{D}) = \{(u^+, v^-), (v^+, u^-) : uv \in E(G)\} \cup \{(v^-, v^+) : v \in V(G)\}$. We then assigned edge capacities by setting

$$c(\vec{e}) = \begin{cases} 1 & \text{if } \vec{e} = (v^-, v^+), v \in V(G) \\ \infty & \text{otherwise} \end{cases},$$

and considered flows in the network $(\vec{D}, x^+, y^-, c)$.

To complete the proof, prove the following claim: the minimum capacity of an x^+, y^- cut in $\vec{D}$ is equal to the minimum size of an x, y-separating set in G.

Exercise 5 In this exercise you have the opportunity to perform the calculations needed in the inductive step in our proof of Baranyai's Theorem. Recall that for $1 \le \ell \le n$, we sought a collection of $M = \binom{n-1}{k-1}$ m-partitions $\mathcal{A}_i$ of $[\ell]$, where $m = \frac{n}{k}$, such that every set $F \subseteq [\ell]$ appeared (with multiplicity) in exactly $\binom{n-\ell}{k-|F|}$ of the m-partitions.

Given such a collection of m-partitions for $\ell \le n-1$, we built a network $(\vec{D}, s, t, c)$, where $V(\vec{D}) = \{s, t\} \cup \{\mathcal{A}_i : i \in [M]\} \cup \{F : F \subseteq [\ell]\}$ and

$$\vec{E}(\vec{D}) = \{(s, \mathcal{A}_i) : i \in [M]\} \cup \{(\mathcal{A}_i, F) : i \in [M], F \in \mathcal{A}_i\} \cup \{(F, t) : F \subseteq [\ell]\}.$$

The capacities were given by

$$c(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F, t) \\ \infty & \textit{otherwise} \end{cases}.$$

(a) Prove that the flow f defined by

$$f(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \frac{k-|F|}{n-\ell} & \vec{e} = (\mathcal{A}_i, F) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F, t) \end{cases}$$

is indeed a feasible flow.

(b) We used an integral maximum flow to find a unique set $F_i \in \mathcal{A}_i$ for each $i \in [M]$, and then formed m-partitions $\mathcal{A}'_i$ of $[\ell + 1]$ by adding the element $\ell + 1$ to the set F_i in each $\mathcal{A}_i$. Show that this collection of m-partitions of $[\ell + 1]$ satisfies the required conditions.

Exercise 6 Suppose $n \geq 2$. Baranyai's Theorem guarantees $\binom{[3n]}{3}$ can be partitioned into perfect matchings without explicitly describing these matchings. In this exercise you will give such an explicit description in the case when $p = 3n - 1$ is a prime number.

(a) Consider the field $\mathbb{F}_p$, and denote by $\mathbb{F}_p^*$ the set of invertible elements, namely $\mathbb{F}_p^* = \{1, 2, \ldots, p-1\}$. Define the map $\pi : \mathbb{F}_p^* \to \mathbb{F}_p$ by $\pi(x) = -(1+x)x^{-1}$. Show that π is injective and $\pi^3(x) = x$ for any $x \neq p - 1$.

(b) Add a new element u to $\mathbb{F}_p$, and extend π to $\{u, 0\}$ injectively so that $\pi^3(x) = x$ for all $x \in \mathbb{F}_p \cup \{u\}$. Show that this gives some perfect matching M_0 in $\binom{[3n]}{3}$.

(c) By considering affine transformations $x \mapsto ax + b$, find another $\binom{3n-1}{2} - 1$ perfect matchings in $\binom{[3n]}{3}$.

(d) Show that these matchings partition $\binom{[3n]}{3}$ into perfect matchings.