Algorithmic Combinatorics | Wi 2016–17
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Exercise Sheet 9

Due date: 14:00, Jan 10th, by the end of the lecture.
Late submissions will be discarded faster than a New Year's resolution.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Consider the linear program given by maximising $\vec{c}^{\,T}\vec{x}$, subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$, where $\vec{c} = (1,1,1,1)^T$, $\vec{b} = (5,4)^T$, and

$$A = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 2 & 2 & 1 & -2 \end{pmatrix}.$$

Determine which pairs of columns form bases, which of those bases are feasible, and solve the linear program.

Exercise 2 Given a graph $G = (V, E)$, a *vertex cover* is a set $S \subseteq V$ such that for every edge $e \in E$, we have $e \cap S \neq \emptyset$. One would like to find the smallest vertex cover of G.

(a) Formulate an integer linear program to solve this problem, and prove rigorously that the solutions to your program correspond to smallest vertex covers of G.

(b) Assuming linear programs can be solved in polynomial time, prove that the linear programming relaxation[1] gives a 2-approximation algorithm for the size of the smallest vertex cover.

Exercise 3 We have defined basic feasible solutions for linear programs in equational (standard) form. For an n-variable linear program in general form, where constraints can be a mix of equalities and inequalities, and some variables may be allowed to take negative values, a basic feasible solution is defined as a feasible solution where n linearly independent constraints[2] are satisfied with equality.

Prove that, for linear programs in equational form, this definition is equivalent to the one we had in lecture.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S09.html`.]

[1] If the relaxation of your solution to (a) is not suitable, then you might want to find another solution.
[2] A set of constraints is linearly independent if the vectors of their coefficients are linearly independent.

Exercise 4 We are given a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$.[3] We also have a start vertex $s \in V$ and an end vertex $t \in V$, and have to find a path of minimum total weight from s to t. Provide an integer linear program to find such a path, and prove that it is indeed a correct formulation of the problem.

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[3] "And just who is handing us these edge-weighted graphs?" Your query, I imagine, is born from having, through personal experience, learnt the most important lesson there is to learn in life[4]: be wary of accepting items from strangers. Let me then spend a few minutes reassuring you as to the provenance of this graph, so that we may proceed with the problem at hand.

The distributor of the weighted graph in question is none other than Monsieur Arsène Wenger, the long-serving and seemingly ever-beleaguered manager of Arsenal FC. The aforementioned graph, meanwhile, is not just some arbitrary graph, but the British rail network. The start vertex is London, home to Wenger's football club, and the end vertex is Bournemouth, where Arsenal are due to play their third league fixture in a span of eight days as part of the Premier League's annual assault on the Christmas season. The weights on the edges represent the cost of travel.

"All this is well and fine," you grumble, "but why are *we* being given this graph? Surely an institution as prosperous as a football club — and especially one that is flush with cash following the signing of a record-breaking TV rights deal — can afford to hire staff to sort out their travel arrangements."

Your grumbles are well-founded, and indeed all other clubs in Arsenal's position would in fact have a private coach[5] to transport the players and support staff from city to city, but Arsenal FC is a club built on grinding out top-four finishes,[6] complaining about inflation in the transfer market, and paying rich dividends to its shareholders. As such, the club usually requires its players to cycle to their away fixtures, arguing that not only does this save costs, but provides a free training[7] session for the players as well. However, the club's board of directors concedes that cycling is impractical during this busy festive season, and thus allows the team to travel by train.

You might think the prospect of a relatively relaxing train ride, as opposed to a gruelling trek by bicycle, would leave Monsieur Wenger in high spirits, but this was not the case. Not only did he not relish the thought of riding to a footballing event with members of the general public,[8] but he knew that unless he found the absolute cheapest route to the match, he would soon be out of a job. Hence he turns to you for help; he has an integer linear program solving machine,[9] but needs you to convert the problem into an instance of an integer program to feed into his machine. Will you assist the poor man?

[4] To be distinguished from the more important lessons available in mathematical courses such as ours, which are supported on a higher plane of existence than this thing we call life.

[5] In the vehicular sense, not the managerial one.

[6] So that they can inevitably be eliminated from the Champions' League by either Barcelona or Bayern München.

[7] In the fitness sense, not the vehicular one.

[8] It is not that Monsieur Wenger despised the common man, but understandably found overhearing the shallow analysis of the many armchair pundits[10] difficult to bear after a difficult day's work. Furthermore, no respite was to be had in the privacy of the train's toilets, inevitably defaced with hurtful graffiti.[11]

[9] Known to the rest of the world by the name Mesut Özil.

[10] Person 1: "Did you see that ludicrous display last night?"

Friend of person 1 (or perhaps a stranger whose acquaintance Person 1 has only just made): "What was Wenger thinking sending Walcott on that early?"

Person 1 again: "The thing about Arsenal is, they always try to walk it in!"

[11] "There once was a manager of Arsenal,
A specialist in failure he was called,
Won nothing for ten years,
But he had no reason to fear,
For profits mattered more than football."

Exercise 5 A professor[12] is computing the final homework grades for the 25 students, $S_1, S_2, \ldots, S_{25}$, in her course. There were a total of 12 exercise sheets, $E_1, E_2, \ldots, E_{12}$, each graded out of 20 points, and for $1 \leq i \leq 25$ and $1 \leq j \leq 12$, student S_i scored $p_{i,j}$ points on exercise sheet E_j. Each student needs to get an average score of 12 points to pass.

However, the professor has a trick up her sleeve — she never said that the assignments would carry equal weight. She is free to choose non-negative weights w_j, $1 \leq j \leq 12$, that will be used to average the students' grades. There is a subset $\mathcal{L} \subseteq \{S_1, S_2, \ldots, S_{25}\}$ of the students that she likes, and she wants to make sure they will pass. On the other hand, there is a disjoint subset $\mathcal{D} \subseteq \{S_1, S_2, \ldots, S_{25}\}$ of students she dislikes, and she wants to ensure they will fail.[13]

Set up, with explanation, a linear program to help her find suitable weights w_j.

Bonus (5 pts) After a change of heart, the professor decides to forgive the students in $\mathcal{D}$, and treat all students equally.[14] Her only objective now is to choose weights so that the maximum possible number of students pass. Provide a mixed linear program to find weights that will achieve this goal.

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[12]Coincidentally enough, Arsène Wenger[3] was not the only disgruntled passenger of the Bournemouth–London Regional Express. Also aboard this train[15] was none other than the curiously but perhaps aptly named Professor Professor,[16] who was not altogether delighted about having to finish the grading for her course, *Politicising Beyoncé*.[17] Still, the sooner she completed her grading, the sooner she could return to the algorithmic research that she so loved,[18] and this motivation, coupled with the slowness of the seemingly sarcastically named Regional Express, saw her complete the onerous task at hand before the train had reached its destination.

[13]This may seem needlessly draconian, even accounting for the natural frustration an instructor would feel at having to teach a course so far from her discipline of choice, but Professor Professor had some very rude and disruptive students. Consider the following comment from the teaching evaluations:

"There once was a professor named Professor,
A hopeless instructor, there can be none lesser,
She doesn't get Bey,
What else can I say,
The university should simply dismiss her."

[14]We shall never know whether this change in policy was born of a seasonal bout of mercy, or the realisation that students she fails may return to retake her course in the future.

[15]Although our two protagonists shared this train, they were sitting in different compartments, and so they remained oblivious to the fact that they were fellow passengers.

[16]Professor Professor's surname caused no end of administrative confusion as she was rising through the academic ranks, and so it was of great relief — indeed, far greater relief than one normally experiences — when she earned tenure. As a postdoctoral fellow, she would have to explain to countless university administrations that she was not a professor, but a Professor, and at times she considered how much easier life would be if she pursued a different profession. However, she stayed the course, if only because she was so amused by how much worse things could be for her future progeny: Junior-Professor Professor Jr.

[17]You may feel that I am now asking too much of your suspension of disbelief, and that this could not possibly be the truth, and yet: `http://www.politicizingbeyonce.com/`.

[18]One could ask why a professor teaching a course about Beyoncé would be interested in algorithmic research, or why one interested in algorithmic research would be asked to teach a course on Beyoncé, and these are very good questions. Indeed, one of the driving forces behind Professor Professor's research was to develop better course scheduling algorithms that would result in more appropriate teaching assignments.

Exercise 6 The recently-hired manager of a newly-formed football team is tasked with buying enough players to form a competitive squad.[19] The manager has a total budget of €114.615.000, with which she must sign a total of 15 players, including at least one goalkeeper, a minimum of 4 defenders, at least 4 midfielders and at least 2 forwards.

The players available on the transfer market are $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, with a partition $\mathcal{P} = \mathcal{G} \sqcup \mathcal{D} \sqcup \mathcal{M} \sqcup \mathcal{F}$ into the sets of goalkeepers, defenders, midfielders and forwards respectively. The function $p : \mathcal{P} \to \mathbb{R}$ gives the price of each player, while $s : \mathcal{P} \to \mathbb{R}$ rates the skill of each individual player.

(a) Write an integer linear program that determines the maximum total skill possible in an affordable squad of 15 players.

(b) Does the linear programming relaxation give any guaranteed approximation of this maximum total level of skill?

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[19]It may surprise you to learn that this manager is none other than Professor Professor,[12] who made a rather remarkable transition from academia to football management. Having completed her grading, Professor Professor returned to her algorithmic work. Desperate to avoid having to ever again teach Beyoncé-related courses, she made several breakthroughs in the field of scheduling and operations research. As a consequence of her work, she discovered a polynomial-time algorithm for three-colourability, thus showing that $\mathcal{P} = \mathcal{NP}$.[20]

This unexpected breakthrough was celebrated throughout the mathematical community, and brought Professor Professor a great deal of deserved recognition. This recognition was not only in the form of laypeople occasionally stopping her on the street, saying that she looked somewhat familiar,[21] but was also realised monetarily. Since she had solved one of the Millenium Problems, she was awarded a million dollars by the Clay Institute. She made further history by winning no fewer than three Nobel Prizes,[22] and picked up the Fields Medal, Abel Prize, Shaw Prize and Breakthrough Prize as well. Most importantly, the dean of her university promised her that she would not have to teach *Politicising Beyoncé* in the near future.

However, all was not well with the world. Now that $\mathcal{P} = \mathcal{NP}$ was known to hold, integer programming could be solved efficiently, and so Mesut Özil's skills[9] were no longer needed by his football club. Ever eager to save costs, Arsenal FC terminated his contract, leaving him unemployed. Bitter about this turn of events, Özil wrote Professor Professor a rather angry letter complaining about his worsened situation. Professor Professor felt distraught, for she had never intended to disrupt anyone else's livelihood. At the same time, having resolved the major open problem in her field, she felt academia had little left to offer her.

She thus resolved to build a football team of her own, which would allow her to sign Özil, and thus make things right by him. Realising she needed a little more capital to establish her team, she got in touch with the manufacturer of the energy drink that had fuelled her research. Spotting an opportunity for some positive publicity, the company were happy to offer to support her, providing €100.000.000 to add to her considerable prize money, and thus leaving us with the problem above.

[20]This part of the problem is fictional. At the time of writing, the $\mathcal{P}$ vs $\mathcal{NP}$ problem remains unsolved, and you are most welcome to work on it.

[21]To say that Professor Professor became a minor celebrity following her breakthrough would perhaps be overstating matters slightly, but she was interviewed by a local newspaper[23] and also featured in an advertisement for a furniture store based in her neighbourhood.

[22]Economics, since the Travelling Salesman Problem was no longer a problem; Literature, for the incredible style in which her article was written; Peace, since the Stable Roommate Problem could now be solved, making domestic disputes a thing of the past.

[23]The article, complete with a photo of our professor, appeared on page 10 of the newspaper, and was seen by roughly 8% of the newspaper's readership of 2174 people.