Algorithmic Combinatorics — Wi 2016–17

Shagnik Das — Tibor Szabó

Exercise Sheet 9

Due date: 14:00, Jan 10th, by the end of the lecture.
Late submissions will be discarded faster than a New Year's resolution.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Consider the linear program given by maximising $\vec{c}^{\,T}\vec{x}$, subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$, where $\vec{c} = (1,1,1,1)^T$, $\vec{b} = (5,4)^T$, and

$$A = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 2 & 2 & 1 & -2 \end{pmatrix}.$$

Determine which pairs of columns form bases, which of those bases are feasible, and solve the linear program.

Exercise 2 Given a graph $G = (V, E)$, a *vertex cover* is a set $S \subseteq V$ such that for every edge $e \in E$, we have $e \cap S \neq \emptyset$. One would like to find the smallest vertex cover of G.

(a) Formulate an integer linear program to solve this problem, and prove rigorously that the solutions to your program correspond to smallest vertex covers of G.

(b) Assuming linear programs can be solved in polynomial time, prove that the linear programming relaxation[1] gives a 2-approximation algorithm for the size of the smallest vertex cover.

Exercise 3 We have defined basic feasible solutions for linear programs in equational (standard) form. For an n-variable linear program in general form, where constraints can be a mix of equalities and inequalities, and some variables may be allowed to take negative values, a basic feasible solution is defined as a feasible solution where n linearly independent constraints[2] are satisfied with equality.

Prove that, for linear programs in equational form, this definition is equivalent to the one we had in lecture.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S09.html`.]

[1] If the relaxation of your solution to (a) is not suitable, then you might want to find another solution.
[2] A set of constraints is linearly independent if the vectors of their coefficients are linearly independent.

Exercise 4 We are given a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$. We also have a start vertex $s \in V$ and an end vertex $t \in V$, and have to find a path of minimum total weight from s to t. Provide an integer linear program to find such a path, and prove that it is indeed a correct formulation of the problem.

Exercise 5 A professor is computing the final homework grades for the 25 students, $S_1, S_2, \ldots, S_{25}$, in her course. There were a total of 12 exercise sheets, $E_1, E_2, \ldots, E_{12}$, each graded out of 20 points, and for $1 \le i \le 25$ and $1 \le j \le 12$, student S_i scored $p_{i,j}$ points on exercise sheet E_j. Each student needs to get an average score of 12 points to pass.

However, the professor has a trick up her sleeve — she never said that the assignments would carry equal weight. She is free to choose non-negative weights w_j, $1 \le j \le 12$, that will be used to average the students' grades. There is a subset $L \subseteq \{S_1, S_2, \ldots, S_{25}\}$ of the students that she likes, and she wants to make sure they will pass. On the other hand, there is a disjoint subset $\mathcal{D} \subseteq \{S_1, S_2, \ldots, S_{25}\}$ of students she dislikes, and she wants to ensure they will fail.

Set up, with explanation, a linear program to help her find suitable weights w_j.

Bonus (5 pts) After a change of heart, the professor decides to forgive the students in $\mathcal{D}$, and treat all students equally. Her only objective now is to choose weights so that the maximum possible number of students pass. Provide a mixed linear program to find weights that will achieve this goal.

Exercise 6 The recently-hired manager of a newly-formed football team is tasked with buying enough players to form a competitive squad. The manager has a total budget of €114.615.000, with which she must sign a total of 15 players, including at least one goalkeeper, a minimum of 4 defenders, at least 4 midfielders and at least 2 forwards.

The players available on the transfer market are $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, with a partition $\mathcal{P} = \mathcal{G} \sqcup \mathcal{D} \sqcup \mathcal{M} \sqcup \mathcal{F}$ into the sets of goalkeepers, defenders, midfielders and forwards respectively. The function $p : \mathcal{P} \to \mathbb{R}$ gives the price of each player, while $s : \mathcal{P} \to \mathbb{R}$ rates the skill of each individual player.

(a) Write an integer linear program that determines the maximum total skill possible in an affordable squad of 15 players.

(b) Does the linear programming relaxation give any guaranteed approximation of this maximum total level of skill?