Recall: a real-life scenario

A company with a 100 employees has six projects running simultaneously, each having its own leader. Each project leader wants to schedule a one hour project meeting, but since an employee might be part of several projects and each project member should be present at each relevant meeting, the scheduling is problematic.

Administration tries to minimize overall time spent with meetings and requests the project leaders to be available between 8-10. Then they try to schedule a conflict-free project meeting schedule by finding a *proper coloring* of the conflict graph of the projects, using the timeslots 8-9 and 9-10 as colors.

Recall: Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring.

Proposition $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic; Greedy coloring (Order vertices arbitrarily; color in this order with first available color)

A more complicated real-life scenario

Project leaders are not so flexible to be available at wish of administration, they want to identify the possible one-hour-slots themselves. One might want to be available 8-10, the other 9-11, the third one 8-9 and 10-11, etc.

Is a conflict-free scheduling still possible? Or the administration should ask project leaders to be available for more than just two one-hour timeslots? How many should they ask for?

This scenario, when each vertex (project) has its own set of available colors (timeslots) is the setting of *list coloring*.

List Coloring

$v \in V(G)$, $L(v)$ a list of colors
A list coloring is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$.

G is k-choosable or k-list-colorable if **every** assignment of k-element lists permits a proper coloring.

$\chi_l(G) = \min\{k : G \text{ is } k\text{-choosable}\}$

Claim $\chi_l(G) \geq \chi(G)$

Example: $K_{2,2}$

Example: $\chi_l(K_{3,3}) \neq \chi(K_{3,3})$

Claim $\chi_l(G) \leq \Delta(G) + 1$

Example: $\chi_l(G) - \chi(G)$ can be arbitrary large:
Proposition $K_{m,m}$ is not k-choosable for $m = \binom{2k-1}{k}$.

Recall: Four-Color Theorem (Appel-Haken, 1976)
Every planar graph is 4-colorable.
Proof: Very-very long, tedious.

Recall: Five-Color Theorem (Heawood, 1890)
Every planar graph is 5-colorable.
Proof: Proved in Discrete Math I.

Theorem. (Thomassen) Every planar graph is 5-list colorable.

HW. There is a planar graph which is not 4-list-colorable.

Proof of Theorem:
Stronger Statement. Let G be a plane graph with an outer face bounded by cycle C. Suppose that
- two vertices v_1, v_2, $v_1v_2 \in E(C)$ are colored by two different colors,
- the other vertices of C have 3-element lists assigned to them and
- the internal vertices have 5-element lists assigned to them.

Then the coloring of v_1 and v_2 can be extended properly to the whole G using colors from the assigned lists for each vertex.

Proof. W.l.o.g. every face of G is a triangle, except maybe the outer face.

Induction on $n(G)$. For $n(G) = 3$, $G = K_3$, OK.

For $n(G) > 3$, there are two cases.

Case 1. There is a chord $v_i v_j$ of C.
Cut to two smaller graphs along the chord, color first the piece where both v_1 and v_2 lie, then color the other piece.

Case 2. C has no chord.
Designate two colors $x, y \in L(v_3)$ such that they differ from the color of v_2. Color $G - v_3$ by induction, such that x and y are deleted from the lists of $N(v_3)$. Extend the coloring to v_3.

Edge coloring

A k-edge-coloring of a multigraph G is a labeling $f: E(G) \to S$, where $|S| = k$. The labels are called colors; the edges of one color form a color class. A k-edge-coloring is proper if incident edges have different labels. A multigraph is k-edge-colorable if it has a proper k-edge-coloring.

The edge-chromatic number (or chromatic index) of a loopless multigraph G is

$$\chi'(G) := \min\{k : G \text{ is } k\text{-edge-colorable}\}.$$

A multigraph G is k-edge-chromatic if $\chi'(G) = k$.

Examples. K_4, K_5, K_n, $\Delta(G) \leq \chi'(G)$

Theorem. (König, 1916)
For a bipartite multigraph G, $\chi'(G) = \Delta(G)$

Proposition. $\chi'(Petersen) = 4$.

Line graphs and Vizing's Theorem

Line graph $L(G)$: vertex set $V(L(G)) = E(G)$ and edge set $E(L(G)) = \{ef : e \cap f \neq \emptyset\}$

Observations

- $M \subseteq E(G)$ is a matching $\Leftrightarrow$ $M \subseteq V(L(G))$ is an independent set
- $c : E(G) \to [k]$ is a proper edge-coloring of G $\Leftrightarrow$ $c : V(L(G)) \to [k]$ is a proper vertex-coloring of $L(G)$
- Hence $\chi'(G) = \chi(L(G))$, so

$$\begin{aligned} \Delta(G) &\leq \omega(L(G)) \\ &\leq \chi'(G) \leq \Delta(L(G)) + 1 \\ &\qquad\qquad \leq 2\Delta(G) - 1 \end{aligned}$$

Theorem. (Vizing, 1964) For a simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Generalization. If the maximum edge-multiplicity in a multigraph G is $\mu(G)$, then $\chi'(G) \leq \Delta(G) + \mu(G)$

Example. Fat triangle; $\chi'(G) = \Delta(G) + \mu(G)$.

Proof of Vizing's Theorem (A. Schrijver)

Induction on $n(G)$.

If $n(G) = 1$, then $G = K_1$; the theorem is OK.

Assume $n(G) > 1$. Delete a vertex $v \in V(G)$. By induction $G - v$ is $(\Delta(G) + 1)$-edge-colorable.

Why is G also $(\Delta(G) + 1)$-edge-colorable?

We prove the following

Stronger Statement. Let $k \geq 1$ be an integer. Let $v \in V(G)$, such that

- $d(v) \leq k$,
- $d(u) \leq k$ for every $u \in N(v)$, and
- $d(u) = k$ for at most one $u \in N(v)$.

Then
$G - v$ is k-edge-colorable $\Rightarrow$ G is k-edge-colorable.

Proof of the Stronger Statement I

Induction on k (!!!)

For $k = 1$ it is OK.

W.l.o.g. $d(u) = k - 1$ for every $u \in N(v)$, except for *exactly one* $w \in N(v)$ for which $d(w) = k$.

Let $f : E(G - v) \to \{1, \dots, k\}$ be a proper k-edge-coloring of $G - v$, which minimizes*

$$\sum_{i=1}^{k} |X_i|^2.$$

Here $X_i := \{u \in N(v) :\ u \text{ is missing color } i\}$.

*I.e., we choose the coloring so the $|X_i|$s "as equal as possible".

Proof of the Stronger Statement II

Case 1. There is an i, with $|X_i| = 1$. Say $X_k = \{u\}$.

Let $G' = G - uv - \{xy : f(xy) = k\}$.
Apply the induction hypothesis for G' and $k - 1$.

Case 2. $|X_i| \neq 1$ for every $i = 1, \ldots, k$.

Then

$$\sum_{l=1}^{k} |X_l| = 2d(v) - 1 < 2k.$$

So there are colors i with $|X_i| = 0$ and
j with $|X_j| \geq 3$.

Let $H \subseteq G$ be subgraph spanned by the edges of color i and j.
Switch colors i and j in a component C of H, where $C \cap X_j \neq \emptyset$.
This reduces $\sum_{l=1}^{k} |X_l|^2$, a contradiction. □

Edge-List Coloring

List Coloring Conjecture (1985) $\chi'_l(G) = \chi'(G)$

Theorem (Kahn, 1996) $\chi'_l(G) = \chi'(G)(1 + o(1))$
Proof is a difficult, probabilistic argument.

Theorem (Galvin, 1995) For any bipartite graph B,

$$\chi'_l(B) = \chi'(B).$$

Here we prove Galvin's Theorem for $B = K_{n,n}$.
HW: Modify it to work for arbitrary bipartite graphs.

Recall $\chi'(K_{n,n}) = n$.

Reformulation of theorem for $B = K_{n,n}$ (Dinitz Conjecture, 1979) For an $n \times n$ square array, a set of n symbols is given for each cell. Then it is possible to select a symbol for each cell among its symbols, such that no row or column repeats a symbol.

Kernels and list-colorings

A kernel of a digraph D is an independent set $S \subseteq V(D)$, such that for every $v \in V(D) \setminus S$ there is $w \in S$, such that $v\vec{w}$.

Remarks The right-to-left orientation of the edges of a graph according to any ordering of its vertices has a kernel: the class of color 1 in the Greedy coloring.

Not every digraph has a kernel.

A digraph is kernel-perfect if every induced subdigraph has a kernel.

Let $f : V(G) \to \mathbb{N}$ be a function. A graph G is called f-choosable if a proper coloring can be chosen from any family of lists $\{L(v)\}_{v \in V(G)}$ provided $|L(v)| \geq f(v)$ for every $v \in V(G)$.

Kernel-perfect digraphs and choosability:

Lemma (Bondy-Boppana-Siegel) Let D be a kernel-perfect orientation of G. Then G is f-choosable with $f(v) = 1 + d_D^+(v)$.

Kernel-perfect orientation of $L(K_{n,n})$

Theorem (Galvin, 1995) $\chi'_l(K_{n,n}) = \chi'(K_{n,n})$.

Proof. Trivially, $n = \Delta(K_{n,n}) \leq \chi'(K_{n,n}) \leq \chi'_l(K_{n,n})$

Claim 1. There is an orientation D of $L(K_{n,n})$ such that $\Delta^+(D) = n - 1$ and for every $v \in V(K_{n,n})$ the restriction of D to $\{vw : w \in N(v)\}$ is transitive.

Claim 2. Let D be an orientation of $L(K_{n,n})$ such that for every $v \in V(K_{n,n})$ the restriction of D to $\{vw : w \in N(v)\}$ is transitive. Then D is kernel perfect.

Claim 1. + **Claim 2.** + **Lemma** $\Rightarrow$
$L(K_{n,n})$ is f-choosable with $f \equiv \Delta^+(D) + 1 = n$.
$\square$

Orienting $L(K_{n,n})$

Proof of Claim 1.

$M = W = \{0, 1, 2, \ldots, n-1\}$

$E(K_{n,n}) = V(L(K_{n,n})) = \{ij : i \in M, j \in W\}$

$ij \to i'j$ if $i + j > i' + j \pmod{n}$

$ij \to ij'$ if $i + j < i + j' \pmod{n}$

$d^+(ij) = n - 1$ for every $ij \in V(L(K_{n,n}))$

For fixed $j \in W$, incident edges are transitively oriented from the edge $(n-j-1)j$ (the source) towards the edge $(n-j)j$ (the sink), going around modulo n.

For fixed $i \in M$, incident edges are transitively oriented from the edge $i(n-i)$ (the source) towards the edge $i(n-i-1)$ (the sink), going around modulo n.
□

Stable Matchings

Bonnie and Clyde is called an unstable pair if

- Bonnie and Clyde are currently not a couple,
- Bonnie prefers Clyde to her current partner, and
- Clyde prefers Bonnie to his current partner.

A perfect matching (of n woman and n man) is a stable matching if it yields no unstable pair.

Theorem. (Gale-Shapley, 1962) There exists a divorce-free society. More precisely: For any preference rankings of n man and n woman there is a stable matching.

Proof. Algorithmic.

The proof of divorce-free society

Proposal Algorithm (Gale-Shapley, 1962)

Input. Preference ranking by each of n man and n woman.

Iteration.
Each man proposes to the woman highest on his list who has not previously rejected him.

IF each woman receives exactly one proposal, THEN
stop and **report** the resulting matching as *stable*.

ELSE

every woman receiving more than one proposal rejects all of them except the one highest on her list.

Every woman receiving at least one proposal says "maybe" to the most attractive proposal she received.

Iterate.

Theorem. The Proposal Algorithm produces a stable matching.

Concluding kernel-perfectness

Proof of Claim 2.
Given an arbitrary subset $S \subseteq V(D)$, we define appropriate preference lists, such that for any stable matching K, $K \cap S$ is a kernel.

Man $i \in M$ prefers woman $j \in W$ to woman $j' \in W$ if

$$ij \in S, ij' \in S \text{ and } ij \leftarrow ij' \text{ or}$$
$$ij \in S, ij' \notin S \text{ or}$$
$$ij \notin S, ij' \notin S \text{ and } ij \leftarrow ij'$$

This is a preference list (a linear ordering of W), because D restricted to $\{ij : j \in W\}$ is transitive

Woman $j \in W$ prefers man $i \in M$ to man $i' \in M$ if

$$ij \in S, i'j \in S \text{ and } ij \leftarrow i'j \text{ or}$$
$$ij \in S, i'j \notin S \text{ or}$$
$$ij \notin S, i'j \notin S \text{ and } ij \leftarrow i'j$$

This is a preference list (a linear ordering of M), because D restricted to $\{ij : i \in M\}$ is transitive

There goes your kernel

Proposition. $K \cap S$ is a kernel for $L(K_{n,n})[S]$

Proof. K is a matching $\Rightarrow K \cap S$ is independent

Suppose there is $ij \in S \backslash K$ which has no outneighbor in $K \cap S$. Let $ij', i'j \in K$.

Then either $ij' \notin S$, or $ij' \in S$ and $ij \leftarrow ij'$.
In both cases i prefers j to j'.

Similarly either $i'j \notin S$ or $i'j \in S$ and $ij \leftarrow i'j$.
In both cases j prefers i to i'.

Hence ij is an unstable pair, a contradiction. □