Algorithmic Combinatorics - Lecture #29 7.2.2017

I. Two-colourability of hypergraphs

A. Review from last week

1. Topics

a) Randomised algorithms

b) Matrix multiplication verification

c) Polynomial identity testing

d) Application: perfect matchings in graphs.

2. Any questions?

B. Motivation

1. Choice of real-world application

a) Motivating application from last week came in for some criticism

(i) Suppose you haven't reached that stage in life where polynomial store scams are a primary concern

b) Will start a new problem now, and didn't want to make the same mistake

(i) Very useful problem - have collected a number of different real-world applications, and will let you choose which one you would find most interesting.

c) Available options:

~~(i) Astrophysics, and how to avoid the fate of the dinosaurs~~

(i) How the dinosaurs could still be alive if they were better at combinatorics

(ii) How combinatorics won Germany the 2014 World Cup

(iii) How combinatorics could save Brangelina,

(iv) The maths is enough for me, thanks.

2. Dinosaurs (if chosen)

a) Dinosaurs were massive and fearsome beasts that roamed the planet hundreds of millions of years ago

b) Suddenly, 66 million years ago, they almost all disappeared, and now all we are left with are birds.

c) Scientists believe this mass extinction event was caused by a large asteroid colliding with Earth

(i) Huge shockwave sent up a cloud of dust and debris, blocking the sun and causing a long-lasting and deadly winter.

d) Why is this of concern to us?

(i) Recently there have been reports of "near misses": asteroids that have come very close to hitting us.

(ii) Two theories for increased incidences:

- Humans used to be hunter-gatherers: if you wanted to eat, you would have to pursue your prey over long distances. Now we have cars and McDonald's → humans are fatter → more attractive, gravitationally speaking ⇒ more asteroids
- We now have better technology to detect these objects as they fly past.

e) How would we defend ourselves from such an asteroid?

(i) Best plan: fly a crew onto the asteroid, bury a nuclear bomb deep within it, and blow it into small harmless pieces.

f) ~~[illegible]~~ Whither combinatorics?

(i) Very dangerous and risky mission

(ii) Need a successful launch of the spacecraft

(iii) Need to safely land said spacecraft on asteroid.

(iv) Cannot afford to rely on just one shuttle; need to send two.

g) Dividing the crew

(i) Successful mission requires many experts

(ii) Need to divide them between two shuttles

(iii) Prepare for the worst: if only one of the shuttles arrives, it should have enough experts to save the planet.

(iv) Different roles required: nuclear expert, drilling expert, communications, leader, astronaut, astrophysicist, biologist, president.

(v) Given the set of people qualified for these roles — hopefully have multiple people for each role, but could also have one person capable of many roles — how can we assign them to the red/blue shuttle so that each shuttle is equipped for success?

2′. World Cup (if chosen)

a) In 2014, the football World Cup ~~[illegible]~~ was held in Brazil, and won by Germany

b) Back when I was a child, there was no way a European team would win a South American tournament: weeks at sea crossing the ocean, seasickness, lack of training ⟹ loss.

(i) Fun aside: in the good old days, cricket used to have timeless Tests: keep playing until one team won.
(ii) Last timeless Test played in 1939, when England toured South Africa.
(iii) England, set a target of 696 to win, were 654/5 after nine days' play, only hours from victory.
(iv) However, they were about to miss their boat home → match was declared a draw.
(v) ~~Nowadays~~ Now, for commercial reasons, Test matches are restricted to five days.

c) Nowadays, of course, we have planes, so no such handicap for European teams

d) Indeed, Germany went on to win the tournament. What was the main reason?
   (i) Was it the defensive stability? (Neuer)
   (ii) Was it the offensive firepower? (7-1)
   (iii) Was it the strength of squad? (Götze)
   (iv) No, it was the careful combinatorial planning before leaving Germany.

e) Bermuda triangle.
   (i) Flying from Germany to Brazil requires travelling over the treacherous Bermuda triangle (draw a map that makes this look feasible)
   (ii) If the whole team flew together on a single plane, there would be a good chance that the plane would

disappear, and the team would not make it to Brazil

(iii) To improve their chances, Die Mannschaft decide to travel on two separate planes, ~~f) Partitioning the~~ and hope that at least one plane makes it.

f) Partitioning the team

(i) Of course, to win a World Cup requires a lot of different roles

(ii) Need goalkeeper, defenders, midfielders, forwards, captain, manager, physio, coaches sports psychologists, PR manager, etc.

(iii) Therefore should ensure that each plane has all the successful ingredients

(iv) Hopefully have multiple people for each role

(v) Could also have people filling multiple roles,
eg Neuer = GK + defender,
Lahm = everything,
Müller = doesn't know what he should be,

(vi) Given the people and the roles they can fill, can we successfully assign them to the red / blue planes such that each plane carries a World Cup-winning squad?

2." Brangelina (if chosen)

a) We have, of course, studied how stable matchings can lead to a divorce-free society

b) Unfortunately, people tend not to listen to us mathematicians when we tell them how to lead their love ~~to~~ lives. $\Rightarrow$ divorce still happens

c) The entire world was delighted when, in 2014, Brad Pitt and Angelina Jolie – both Hollywood

superstars – got married (to each other)
(i) Considered the world's most glamorous pair since Brad Pitt + Jennifer Aniston.

d) However, the world shook when last year they filed for divorce
(i) No doubt historians will look back at this as being the worst thing to happen in 2016.

e) Still, does not have to be the end of the world – hopefully they can negotiate an amicable split, and continue to share a positive, friendly relationship
(i) After all, the world needs Mr and Mrs Smith 2.

f) Dividing possessions
(i) Brangelina need to find a way to divide their joint possessions in such a way that both feel they get a fair deal ⇒ no resentment
(ii) Have a lot of types of possessions to split: mansions, children, pets, stacks of thousand dollar bills, toasters, etc.
(iii) Different categories of things that they each want: something valuable, a son, a daughter, a loyal companion, a cherished keepsake from their relationship, something easy to transport, etc. etc.
(iv) Need to have multiple things in each category
(v) Single items could also fall in multiple

categories, eg. son could be easy to transport, while a toaster could be a cherished keepsake (if wedding gifts)

(ii) Problem: given the list of items, and the categories that each party should have filled, can an agreeable division be found?

C. Mathematical formulation.

1. Hypergraph

a) Can model these problems combinatorially

b) X = set of items to be divided

c) $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ ~~represent~~ a family of subsets of X, representing the different categories of items.

2. Colouring

a) Our goal: divide the items into two sets, such that each set has at least one item from each category.

b) Can realise this through colouring: colour elements in X either red or blue

(i) Goal: avoid a monochromatic F_i: where all elements are the same colour

c) If such a colouring exists, we say $\mathcal{F}$ is two-colourable (or has Property B)

3. Uniformity

a) Clearly if $|F_i| \leq 1$ for some i, then $\mathcal{F}$ is not two-colourable $\Rightarrow$ need $|F_i| \geq 2$ $\forall i$.

b) If $|F_i| = 2$ $\forall i$, can think of $\mathcal{F}$ as the edges of a graph:

(i) two-colourable $\Leftrightarrow$ bipartite.

(ii) easy to determine when a graph is bipartite.

c) The larger the sets F_i, the easier $\mathcal{F}$ should be to 2-colour

d) $\Rightarrow$ usually restrict to the k-uniform setting: assume $|F_i| = k \ \forall i$, some $k \geq 2$.

D. Randomised algorithm.

1. Complexity

a) Unfortunately, determining whether a k-uniform hypergraph is two-colourable is NP-complete for all $k \geq 3$.

b) $\Rightarrow$ do not expect to find an efficient algorithm to solve this problem

c) However, can try <u>randomised algorithms</u>

2. Uniform colouring

a) Natural attempt: colour the elements of X randomly, and then check if this gives a valid two-colouring.

(i) Symmetry between red/blue $\Rightarrow$ colour with probability $\frac{1}{2} - \frac{1}{2}$

(ii) Simplest model: colour elements independently of one another.

3. Analysis.

a) <u>Claim</u>: If $\mathcal{F} = \{F_1, \dots, F_m\}$ is a k-graph, the probability of the uniformly random 2-colouring containing a monochromatic set is at most $m \cdot 2^{1-k}$.

b) <u>Proof</u>:

(i) For $1 \leq i \leq m$, let
$$E_i = \{\text{the set } F_i \text{ is monochromatic}\}$$

(ii) Then
$$\mathbb{P}(E_i) = \underbrace{2}_{\text{red or blue}} \cdot \underbrace{2^{-k}}_{\text{prob. that all } k \text{ vxs choose this colour.}} = 2^{1-k}$$

(iii) Hence

$$\mathbb{P}(\exists \text{ m.c. set}) = \mathbb{P}\left(\bigcup_{i=1}^{m} E_i\right)$$

$$[\text{union bound}] \leq \sum_{i=1}^{m} \mathbb{P}(E_i)$$

$$= \sum_{i=1}^{m} 2^{1-k} = m\,2^{1-k}. \quad \square$$

E. Extremal problem

1. Small hypergraphs

a) Effective bound

(i) Note that $m 2^{1-k} < 1 \iff m < 2^{k-1}$

(ii) $\Rightarrow \mathbb{P}(\text{good colouring}) > 0$ if $m < 2^{k-1}$

b) Existence of good colouring

(i) $\Rightarrow$ If $m < 2^{k-1}$, then $\mathcal{F}$ must have a two-colouring $\Rightarrow \mathcal{F}$ is two-colourable

(ii) We have proven the <u>existence</u> of a two-colouring for small enough hypergraphs, without explicitly finding one.

(iii) $\Rightarrow$ if asked if $\mathcal{F}$ is two-colourable, and $|\mathcal{F}| < 2^{k-1}$, we can immediately return YES, and will be correct.

2. Improving the bound

a) Is this tight?

(i) In our proof, we used the union bound $\Rightarrow$ used an inequality

(ii) Can we get a better bound on m that guarantees two-colourability?

b) Existence of non-two-colourable k-graphs.

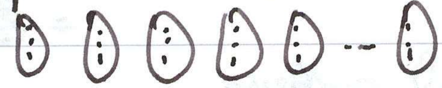
(i) <u>Def^n</u>: Let $m(k) = \min\{|\mathcal{F}| : \mathcal{F}$ is a k-graph that is <u>not</u> 2-colourable$\}$

(ii) We have shown $m(k) \geq 2^{k}$.

(iii) <u>Q</u>: What is $m(2)$? <u>A</u>: 3. $\triangle$

(iv) <u>Q</u>: Is $m(k)$ finite? or when k is larger, do we have enough room to always two-colour any k-graph?

(v) <u>Remark</u>: There are certainly large k-graphs that <u>are</u> two-colourable:

eg: 

$\Rightarrow$ problem is to find <u>small</u> k-graphs that are <u>not</u> 2-colourable.

c) A first upper bound

(i) ~~Claim: $m(k) \leq$~~ Q: Why is Δ not 2-colourable?

(ii) <u>A</u>: In any colouring, two vxs get the same colour, and these form a m.c. edge.

(iii) Can generalise this construction to larger k.

(iv) <u>Claim</u>: $m(k) \leq \binom{2k-1}{k}$

(v) <u>Proof</u>: • Consider $\mathcal{F} = \binom{[2k-1]}{k}$, so $2k-1$ vertices, and every k-set is an edge

• In any 2-colouring, some colour has $\geq k$ vertices [pigeonhole]

• $\Rightarrow$ we get a monochromatic edge.

• $\Rightarrow \mathcal{F}$ is not 2-colourable

• $\therefore m(k) \leq |\mathcal{F}| = \binom{2k-1}{k}$. $\square$.

d) <u>Remarks</u>:

(i) $\binom{2k-1}{k} \leq 4^k$, so we know $m(k)$ is exponential.

(ii) By now, better bounds are known:

• $m(k) = O(k^2 2^k)$ [see HW]

• $m(k) = \Omega\left(\sqrt{\frac{k}{\ln k}}\, 2^k\right)$ [see Extremal Combinatorics]

II. Maker-Breaker Games

A. Derandomisation

1. Return to an algorithmic viewpoint
   a) Given a small k-graph – say with fewer than 2^{k-1} edges – we know there exists a 2-colouring.
   b) Can we explicitly find one?
   c) Know 2-colourability is NP-hard in general, but perhaps finding a proper colouring is easier when the decision problem is trivial.
2. General framework.
   a) Start with a probabilistic argument to give an existence result
      (i) This proves the answer is Yes/No, without giving a certificate.
   b) In some cases, we can convert this into a randomised algorithm
      (i) This is an algorithm using random bits that will, with decent probability, produce a certificate
      (ii) Note: in some cases, like our two-colourability claim, the probabilistic argument may already be algorithmic, but sometimes (eg next week) it is just existential.
   c) Final step: derandomise: find a deterministic algorithm to produce a certificate
      (i) For critical applications, may not be able to wait for randomised algorithm to complete (Las Vegas) or afford the possibility of an error (Monte Carlo).

3. Complexity concerns
   a) Open problem in complexity theory: can every randomised algorithm be efficiently derandomised, or does randomness actually help?
   b) Consensus (perhaps): randomness <u>does</u> help.
   c) Still, in some cases, we have been able to derandomise algorithms.

B. Maker-Breaker set-up
   1. Deterministic colouring
      a) Suppose you are given a k-graph $\mathcal{F}$ to find a proper two-colouring of
      b) Suppose further that you have a partner to help you.
      c) One approach: you colour vxs red, and your partner colours vxs blue.
      d) Goals:
         (i) Your goal: every edge of $\mathcal{F}$ should have a red vx.
         (ii) Partner's goal: every edge of $\mathcal{F}$ should have a blue vx.
         (iii) If both succeed, we get a proper two-colouring ✓.
      e) Given symmetry between red, blue, no reason one should appear more often than another ⇒ take turns, colouring one at a time.
   2. Antagonistic partner
      a) Can we find a strategy where both parties achieve their goals?

b) To answer this, we solve a harder problem (!): can you achieve your goal in the worst case?

(i) Worst-case: partner does everything possible to try and stop you from achieving your goal.

c) New goals:

(i) Your goal: ensure every edge in $\mathcal{F}$ has a red vertex

(ii) Partner's goal: prevent you from achieving your goal; i.e. have some edge of $\mathcal{F}$ that only has blue vertices.

(iii) New names: you are "Breaker"; partner is "Maker".

3. Winning graphs

a) Terminology

(i) Say $\mathcal{F}$ is "Breaker's Win" iff Breaker has a strategy for always achieving his goal, regardless of what Maker does.

(ii) Say $\mathcal{F}$ is "Maker's Win" if Maker has a strategy for always achieving her goal, no matter what Breaker does.

b) Remarks

(i) We assume for now that Maker starts.

(ii) The smaller $\mathcal{F}$ is, the harder it is for Maker: Breaker just needs one vx from each edge, while Maker needs all k vxs from some edge.

c) Extremal problem

(i) Def^n: Let $\tilde{m}(k) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \underline{\text{not}} \text{ Breaker's Win}\}$.