Algorithmic Combinatorics - Lecture #30

8.2.2017

I. Deterministic Two-Colouring

A. Review

1. Extremal problem
   a) Def^n: $m(k) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is not two-colourable}\}$
   b) Claim: $m(k) \geq 2^{k-1}$.
      (i) $\Rightarrow$ if $|\mathcal{F}| < 2^{k-1}$, then $\mathcal{F}$ is two-colourable
      (ii) proof is existential; does not show us how to colour $\mathcal{F}$.
2. Maker-Breaker game
   a) Cooperative players
      (i) Take turns, colouring one element at a time
      (ii) Partner colours elements blue, goal: every $F \in \mathcal{F}$ should have something blue
      (iii) You colour elements red, goal: every $F \in \mathcal{F}$ should have something red.
      (iv) If successful $\rightsquigarrow$ (balanced) proper two-colouring.
      (v) What should your strategies be?
   b) Antagonistic players
      (i) We solve a harder problem: find strategies that work even in the worst case.
      (ii) Worst case: "partner" actively tries to stop you from achieving your goal.
      (iii) Partner, "Maker", colours elements blue; goal: some $F \in \mathcal{F}$ should be all blue;
      (iv) You, "Breaker", colour elements red. goal: every $F \in \mathcal{F}$ should have a red element.
   c) Extremal problem
      (i) Obs: adding sets to $\mathcal{F}$ makes it easier for Maker, harder for Breaker
      (ii) We assume Maker starts.

(iii) <u>Def^n</u>: Hypergraph $\mathcal{F}$ is '<u>Breaker's win</u>'
if Breaker has a strategy to always achieve his goal, regardless of what Maker does.

(iv) <u>Def^n</u>: Hypergraph $\mathcal{F}$ is <u>Maker's win</u>
if Maker has a strategy to always achieve her goal, no matter what Breaker does.

(v) $\tilde{m}(k) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \underline{\text{not}} \text{ Breaker's win}\}$.

B. Relation to two-colourability

1. Statement
   a) <u>Prop^n</u>: For all $k \geq 2$, $m(k) \geq \tilde{m}(k)$.

2. Proof
   a) We need to show that if $|\mathcal{F}| < \tilde{m}(k)$, then $\mathcal{F}$ is two-colourable.
   b) $|\mathcal{F}| < \tilde{m}(k) \Rightarrow \mathcal{F}$ is Breaker's win
      (i) ie. Breaker has a strategy to ensure that no matter what his opponent does, he claims one element from each $F \in \mathcal{F}$.
   c) When colouring with our partner, we use Breaker's strategy
      (i) $\Rightarrow$ every $F \in \mathcal{F}$ gets a red element.
   d) Our partner also uses Breaker's strategy!
      (i) $\Rightarrow$ every $F \in \mathcal{F}$ gets a blue element too
      (ii) $\Rightarrow$ get a proper two-colouring of $\mathcal{F}$ ✓
   e) Slight hitch: Breaker's strategy assumes Maker plays first, so how can our

partner, who goes first, use it?

(i) Obs: having an extra element cannot hurt → only helps us achieve our goals

(ii) In the first round, partner chooses some arbitrary element v_0

(iii) She then forgets about it, and pretends that we are going first, and plays according to Breaker's strategy.

(iv) If the strategy ever tells her to select v_0, ~~then~~ (which she already has), then she takes another arbitrary available element, calling that v_0 instead.

(v) In this way she can successfully use Breaker's strategy. □

3. Remarks

a) We will prove that $\tilde{m}(k) \geq 2^{k-1}$, so this strengthens our previous $m(k) \geq 2^{k-1}$ bound

b) Shows that these hypergraphs have balanced proper two-colourings

c) In proving $\tilde{m}(k) \geq 2^{k-1}$, we will give an explicit (and efficient) strategy for Breaker ⇒ this gives a derandomised algorithm for two-colouring small hypergraphs.

II. Positional Games

A. Motivation

1. Context for Maker-Breaker games

a) We introduced Maker-Breaker games as a tool for derandomising two-colouring

b) Terminology may seem confusing: why Maker? Breaker? Game?

c) Maker-Breaker games are central to the theory of positional games

d) Will now place them in this context.

B. Strong games

1. Set-up

a) Consider a game played between First-Player and Second-Player

b) Game played on a hypergraph $(X, \mathcal{F})$

(i) X is called the "<u>board</u>"

(ii) $\mathcal{F} = \{F_1, \ldots, F_m\}$ called the "<u>winning sets</u>"

c) Players take alternate turns, starting with FP, claiming one element from X at a time

d) Winner: first person to claim all elements from some winning set $F_i \in \mathcal{F}$.

(i) If all elements from X are claimed without anyone winning, game is declared a draw.

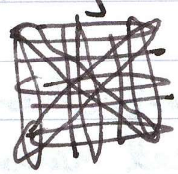
2. Examples

a) Surprisingly, you have been playing positional games for years!

b) Tic-Tac-Toe is a positional game

(i) Board X is the 3×3 grid.

(ii) Winning sets $\mathcal{F}$ are the eight lines:



(iii) (Spoilers) Optimal play $\Rightarrow$ draw.

3. Connection to game theory?
   a) Can think of these as zero-sum two-player games
      (i) Strategies = choice of moves
      (ii) Payoffs: +1 for win, 0 for draw, -1 for loss
   b) Games are "trivial"
      (i) Can be shown [HW?] that strong games are always either FP win or draw
      (ii) ⇒ no "interesting" Nash equilibria; either FP has a strategy that always wins, or SP has a strategy that always draws.
   c) Explicit strategies
      (i) In practice, way too many strategies to compute the payoff matrix
      (ii) Interest is in explicitly determining the optimal strategies.

C. Maker-Breaker games
   1. Second player's objective
      a) Impossible for SP to have a winning strategy
      b) ⇒ Best SP can hope for is to force a draw, i.e. prevent FP from claiming a winning set
   2. Simplified game
      a) Change SP's objective
      b) FP wins if she can claim all elts of a winning set $F_i \in \mathcal{F}$
         (i) Call her <u>Maker</u>
      c) SP wins if he can stop Maker
         (i) ⟺ claiming at least one elt in each winning set
         (ii) call him <u>Breaker</u>.
      d) Related game, but different
         (i) Breaker's win ⇒ draw in strong game.

(ii) However, it doesn't matter now if Breaker claims a full winning set
$\Rightarrow$ Maker doesn't need to "defend", can play pure offence

(iii) eg: Maker-Breaker Tic-Tac-Toe is Maker's win (play game?)

3. Advantages of Maker-Breaker vs Strong games
   a) More amenable to analysis $\Rightarrow$ more results
   b) Deep connections to other areas: extremal, probabilistic combinatorics.

III. The Erdős-Selfridge ~~Condition~~ Criterion

A. Statement

1. Motivation
   a) A very general result which gives a winning strategy for Breaker in many situations
2. Thm (Erdős-Selfridge, 1973)
   Breaker has a winning strategy for the game on $(X, \mathcal{F})$ if
   $$\sum_{F \in \mathcal{F}} 2^{-|F|} < \frac{1}{2}.$$

B. Corollary: uniform case

1. Cor: $\tilde{m}(k) \geq 2^{k-1}$
2. Pf:
   a) Let $\mathcal{F}$ be a k-graph with $|\mathcal{F}| < 2^{k-1}$
   b) Then $\sum_{F \in \mathcal{F}} 2^{-|F|} = 2^{-k} \cdot |\mathcal{F}| < \frac{1}{2}$
   c) Thm $\Rightarrow$ $\mathcal{F}$ is Breaker's win ✓ $\square$

C. Proof

1. Idea
   a) Once Breaker has claimed an element

from a winning set, that set is no longer relevant to the game.

b) Of the sets that remain relevant, the smallest pose the greatest concern for Breaker, as Maker could more quickly claim them

(i) smallest = in terms of number of unclaimed elements

c) Idea: keep track of how "close" Maker is to completing a winning set, and then make choices ~~to try to maximi~~ according to this measure.

2. Danger function

a) Claimed elements

(i) After i turns, let $R_i \subseteq X$ be the elements claimed by Maker, and let $S_i \subseteq X$ be the elements claimed by Breaker.

(ii) Initially, $R_0 = S_0 = \emptyset$, and $R_i \cap S_i = \emptyset \ \forall i$.

b) Winning sets

(i) Breaker need not worry about any winning set $F \in \mathcal{F}$ with $F \cap S_i \neq \emptyset$, since it is impossible for Maker to claim all of F

(ii) If $F \cap S_i = \emptyset$, then Maker has to claim $|F \setminus R_i|$ more elements from F to complete this winning set $\rightarrow$ the smaller $|F \setminus R_i|$, the more dangerous for Breaker.

(iii) Define $\text{danger}_i(F) = \begin{cases} 2^{-|F \setminus R_i|} & \text{if } F \cap S_i = \emptyset \\ 0 & \text{if } F \cap S_i \neq \emptyset \end{cases}$

to represent the danger posed by the winning set $F \in \mathcal{F}$ to Breaker after i turns.

(iv) Globally let $\text{danger}_i(\mathcal{F}) = \sum_{F \in \mathcal{F}} \text{danger}_i(F)$.

c) Vertices

(i) $\text{danger}_i(\cdot)$ helps Breaker keep track of winning sets, but he needs to colour vertices

(ii) Given $x \in X$, let its weight be

$$w_i(x) = \sum_{\substack{F \in \mathcal{F}: \\ x \in F}} \text{danger}_i(F).$$

3. Evolution of danger

a) Changing over time

(i) As the game progresses, the sets R_i and S_i grow

(ii) $\Rightarrow$ the $\text{danger}_i(\cdot)$ function changes, but how?

b) Maker's move

(i) Suppose the $(i+1)^{st}$ move is Maker's, and she chooses $x \in X \setminus (R_i \cup S_i)$.

(ii) If $x \notin F$, then there is no change to the danger function for F.

(iii) If $x \in F$, but $F \cap S_i \neq \emptyset$, then $\text{danger}_i(F) = 0$, and this remains unchanged.

(iv) If $x \in F$ and $F \cap S_i (= F \cap S_{i+1}) = \emptyset$, then $F \setminus R_{i+1} = (F \setminus R_i) \setminus \{x\}$, and $|F \setminus R_{i+1}| = |F \setminus R_i| - 1$, so the danger doubles.

(v) Summary: danger doubles if $x \in F$, and is otherwise unchanged.

(vi) $\Rightarrow \text{danger}_{i+1}(\mathcal{F}) = \sum_{\substack{F \in \mathcal{F}: \\ x \notin F}} \text{danger}_i(F) + \sum_{\substack{F \in \mathcal{F}: \\ x \in F}} 2\,\text{danger}_i(F)$

$= \sum_{F \in \mathcal{F}} \text{danger}_i(F) + \sum_{F \in \mathcal{F}: x \in F} \text{danger}_i(F)$

$= \underline{\text{danger}_i(\mathcal{F}) + w_i(x)}.$

c) Breaker's move

(i) Suppose now that the $(i+1)^{st}$ move is Breaker's, and he plays $x \in X \setminus (R_i \cup S_i)$

(ii) If $x \notin F$, then danger of F is the same

(iii) If $x \in F$, then danger of F becomes 0.

(iv) $\Rightarrow \text{danger}_{i+1}(\mathcal{F}) = \sum_{\substack{F \in \mathcal{F}: \\ x \notin F}} \text{danger}_i(F)$

$$= \sum_{F \in \mathcal{F}} \text{danger}_i(F) - \sum_{\substack{F \in \mathcal{F}: \\ x \in F}} \text{danger}_i(F)$$

$$= \underline{\text{danger}_i(\mathcal{F}) - w_i(x)}$$

4. Breaker's strategy

a) Heuristic: danger is bad

(i) $\Rightarrow$ Breaker should choose vertices to minimise the danger of $\mathcal{F}$.

b) $\Rightarrow$ Strategy: If turn i is Breaker's turn, then Breaker should choose an element $x \in X \setminus (R_{i-1} \cup S_{i-1})$ of maximum weight (as this will minimise $\text{danger}_i(\mathcal{F})$).

c) This is a winning strategy

(i) Claim: With this strategy, and assuming the Erdős-Selfridge Criterion is satisfied,

$\text{danger}_i(\mathcal{F}) < 1$ for all $i \geq 0$.

(ii) Now suppose this was not a winning strategy for Breaker

(iii) $\Rightarrow$ Maker can at some point fully claim a winning set $F \in \mathcal{F}$.

(iv) $\Rightarrow \exists i$ s.t. $F \subseteq R_i \Rightarrow \text{danger}_i(F) = 2^{-|F \setminus R_i|} = 1$

~~⇒ danger~~

(v) $\Longrightarrow \text{danger}_i(\mathcal{F}) \geq \text{danger}_i(F) = 1$ #

5. Bounded danger (proof of Claim)

a) Erdős-Selfridge Criterion

$\Rightarrow \text{danger}_0(\mathcal{F}) = \sum_{F \in \mathcal{F}} 2^{-|F|} < \frac{1}{2}$ ✓

b) First move is Maker's; suppose she chooses $x_1 \in X$.

(i) $\Rightarrow \text{danger}_1(\mathcal{F}) = \text{danger}_0(\mathcal{F}) + w_0(x_1)$
$\leq 2\,\text{danger}_0(\mathcal{F}) < 1$ ✓

c) Now suppose we have $\text{danger}_i(\mathcal{F}) < 1$, and the $(i+1)^{st}$ turn is Breaker's, for some $i \geq 1$.

(i) Suppose Breaker chooses x_{i+1}, and Maker chooses x_{i+2} in her subsequent turn.

(ii) $\text{danger}_{i+1}(\mathcal{F}) = \text{danger}_i(\mathcal{F}) - w_i(x_{i+1})$
$\leq \text{danger}_i(\mathcal{F}) < 1$ ✓

(iii) $\text{danger}_{i+2}(\mathcal{F}) = \text{danger}_{i+1}(\mathcal{F}) + w_{i+1}(x_{i+2})$
$= \text{danger}_i(\mathcal{F}) - w_i(x_{i+1}) + w_{i+1}(x_{i+2})$

(iv) Since ~~Maker's move~~ Breaker's move can only decrease the danger of winning sets, we must have $w_{i+1}(x) \leq w_i(x)$ for all $x \in X$.

(v) $\Rightarrow \text{danger}_{i+2}(\mathcal{F}) \leq \text{danger}_i(\mathcal{F}) - w_i(x_{i+1})$
$+ w_i(x_{i+2})$.

(vi) Breaker's strategy: choose heaviest free vertex $\Rightarrow w_i(x_{i+1}) \geq w_i(x_{i+2})$.

(vii) $\Rightarrow \text{danger}_{i+2}(\mathcal{F}) \leq \text{danger}_i(\mathcal{F})$ ✓

□ Claim

□ Theorem

D. Remark

1. Danger function motivation

a) $\text{danger}_i(F) = 2^{-|F \setminus R_i|}$ if $F \cap S_i = \emptyset$

= prob. that Maker will get F if remaining elements are divided uniformly at random

b) $\Rightarrow$ $\text{danger}_i(\mathcal{F})$ = expected # of winning sets Maker will claim if remaining elements are distributed uniformly at random.

c) $\text{danger}_i(\mathcal{F}) < 1 \Rightarrow$ with pos. prob., Maker does not claim any winning set

d) We proved that Breaker can guarantee hes does no worse than in this random assignment.

2. Probabilistic intuition

a) In many positional games the threshold matches what would happen in the random setting

b) $\Rightarrow$ two clever players get the same result as two random players

c) Does <u>not</u> mean that playing randomly is optimal: a clever player will beat a random player.

3. Derandomisation

a) Observe that computation of danger, weight can all be done efficiently

b) $\Rightarrow$ this is an efficient explicit strategy for Breaker

c) $\rightsquigarrow$ derandomised two-colouring of small k-graphs.

4. Any questions?