9.2 The Local Lemma

In this section we discuss a classic result of Erdős and Lovász generalizing the setup of Claim 2.2.1. We will guarantee the 2-colorability of a hypergraph, but instead of the total size, we will require a bound only on the number of edges intersecting any fixed edge.

Let $(X, \mathcal{F})$ be a hypergraph. The degree $\deg(x) = \deg_{\mathcal{F}}(x)$ of a vertex is the number of edges of $\mathcal{F}$ containing x, and $\Delta(\mathcal{F}) = \max\{\deg(x) : x \in V(\mathcal{F})\}$ is the *maximum degree of* $\mathcal{F}$. The *line graph* $L(\mathcal{F})$ is the graph defined on vertex set $V(L(\mathcal{F})) = \mathcal{F}$ with edge set $E(L(\mathcal{F})) = \{ef : e, f \in \mathcal{F}, e \cap f \neq \emptyset\}$. So $\Delta(L(\mathcal{F}))$ represents the maximum, over all edges e, of the number of those edges of $\mathcal{F}$ (distinct from e) which intersect e.

Clearly $\Delta(L(\mathcal{F})) \leq |\mathcal{F}|$. It turns out that a small loss in the constant factor of the assumption of Claim 2.2.1 already allows us to show 2-colorability of a hypergraph for which only $\Delta(L(\mathcal{F}))$ is bounded (instead of $|\mathcal{F}|$). Our rendition here is based on the one in [10].

Theorem 9.2.1 (Erdős-Lovász [29]). *Let $\mathcal{F} \subseteq \binom{X}{k}$ be a k-uniform hypergraph. Then*

$$\Delta(L(\mathcal{F})) \leq 2^{k-3} \Rightarrow \ \mathcal{F} \textit{ is 2-colorable.}$$

Proof. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$. We will apply the same random coloring procedure as for Claim 2.2.1 and color each vertex independently, uniformly at random with `red` or `blue`. Note however the following difficulty: the expected number of monochromatic edges is $\frac{2|\mathcal{F}|}{2^k}$, which could be arbitrarily large, as we have no restriction on the size of $\mathcal{F}$. So, unlike in the proof of Claim 2.2.1, here we cannot conclude anything based on immediate probabilistic considerations.

Of course, if the hypergraph consisted only of pairwise disjoint edges, then all events would be mutually independent and everything would be easy:

$$\begin{aligned} \Pr[\text{no } A_i \text{ is monochromatic}] &= \prod_{i=1}^{m} \Pr[A_i \text{ is properly colored}] = \prod_{i=1}^{m} \left(1 - \frac{2}{2^k}\right) \\ &= \left(1 - \frac{1}{2^{k-1}}\right)^m > 0. \end{aligned} \tag{9.1}$$

Thus a proper coloring would exist. Note that the probability of success is extremely small, but it does not matter, since for the conclusion we only need that it is positive.

We hope to save this argument by capitalizing on the assumption that each edge intersects only a limited number of other edges. This condition assures that dependence between "bad events" (the edges being monochromatic) is limited.

For a subset $I \subseteq [m]$ of the indices we introduce the notation $\texttt{Proper}_I$ to indicate the event that in the random coloring each A_i, $i \in I$, has vertices of both colors and for an integer $j \in [m]$, we let $\texttt{Mono}_j$ denote the event that A_j is fully $\texttt{red}$ or fully $\texttt{blue}$.

First we show that having a proper coloring of the first i sets does not influence too adversely the chances of the $(i+1)$st set being properly colored: the probability of failure is only a multiplicative factor 2 worse than it would be in the case of a completely disjoint (and hence independent) A_{i+1}. The theorem will then follow from this claim relatively easily.

Claim 9.2.2. *For all $i \in [m]$, we have*

$$\Pr\left[\texttt{Proper}_{[i]} \cap \texttt{Mono}_{i+1}\right] \leq \frac{2}{2^{k-1}} \Pr\left[\texttt{Proper}_{[i]}\right].$$

Proof. In order to use induction we prove a seemingly more general formulation: *For all $I \subseteq [m]$ and for all $j \in [m] \setminus I$ we have*

$$\Pr\left[\texttt{Proper}_I \cap \texttt{Mono}_j\right] \leq \frac{2}{2^{k-1}} \Pr\left[\texttt{Proper}_I\right]. \tag{9.2}$$

We prove this statement by induction on $|I|$. For given $I \subseteq [m]$ and $j \in [m] \setminus I$ let $J = J(I, j) \subseteq I$ be the set of those indices $\ell \in I$ for which $A_\ell \cap A_j \neq \emptyset$. Note that by our assumption on the maximum edge neighborhood size, $|J| \leq 2^{k-3}$.

If $J = \emptyset$, then A_j is disjoint from $\bigcup_{i \in I} A_i$ and hence it being monochromatic is independent from anything that happens with the A_i, $i \in I$. So instead of inequality (9.2) we have an equality, with the constant $\frac{2}{2^{k-1}}$ replaced by $\frac{1}{2^{k-1}}$:

$$\Pr\left[\texttt{Proper}_I \cap \texttt{Mono}_j\right] = \frac{1}{2^{k-1}} \Pr\left[\texttt{Proper}_I\right].$$

The base case $|I| = 0$ of our induction is a special case of this as $\emptyset = I \supseteq J$.

Assume now that $I \supseteq J \neq \emptyset$. By the definition of J, A_j is disjoint from $\bigcup_{i \in I \setminus J} A_i$, hence it being monochromatic is independent from whatever happens to the $A_i, i \in I \setminus J$. Thus

$$\frac{\Pr\left[\texttt{Proper}_{I \setminus J} \cap \texttt{Mono}_j\right]}{\Pr\left[\texttt{Proper}_{I \setminus J}\right]} = \frac{1}{2^{k-1}}. \tag{9.3}$$

We want to compare our goal inequality (9.2) to this equality. Using induction we will replace $I \setminus J$ with I and the equality with an inequality while paying only a price of a multiplicative factor 2. For the numerator we just use the trivial set inclusion

$$\texttt{Proper}_I \cap \texttt{Mono}_j \subseteq \texttt{Proper}_{I \setminus J} \cap \texttt{Mono}_j \tag{9.4}$$

to estimate the probabilities. For the denominator we note that

$$\texttt{Proper}_I = \texttt{Proper}_{I \setminus J} \setminus \bigcup_{i \in J} \left(\texttt{Proper}_{I \setminus J} \cap \texttt{Mono}_i\right),$$

and apply a simple union bound for the probabilities. Since $|I \setminus J| < |I|$, the induction hypothesis also applies and we have

$$\begin{aligned}
\Pr\left[\texttt{Proper}_I\right] &\geq \Pr\left[\texttt{Proper}_{I\setminus J}\right] - \sum_{i\in J} \Pr\left[\texttt{Proper}_{I\setminus J} \cap \texttt{Mono}_i\right] \\
&\geq \Pr\left[\texttt{Proper}_{I\setminus J}\right] - \sum_{i\in J} \frac{2}{2^{k-1}} \cdot \Pr\left[\texttt{Proper}_{I\setminus J}\right] \\
&= \Pr\left[\texttt{Proper}_{I\setminus J}\right] \left(1 - \frac{2|J|}{2^{k-1}}\right) \\
&\geq \Pr\left[\texttt{Proper}_{I\setminus J}\right] \cdot \frac{1}{2}. \qquad (9.5)
\end{aligned}$$

The last inequality holds because $|J| \leq 2^{k-3}$. Estimating the numerator and denominator in (9.3) by (9.4) and (9.5), respectively, implies (9.2). □

To complete the proof of Theorem 9.2.1 we apply Claim 9.2.2 repeatedly and obtain a lower bound on the probability of success. Our lower bound will be only slightly worse than what happens in (9.1), the mutually independent case:

$$\begin{aligned}
\Pr\left[\texttt{Proper}_{[m]}\right] &= \Pr\left[\texttt{Proper}_{[m-1]}\right] - \Pr\left[\texttt{Proper}_{[m-1]} \cap \texttt{Mono}_m\right] \\
&\geq \Pr\left[\texttt{Proper}_{[m-1]}\right] \left(1 - \frac{2}{2^{k-1}}\right) \\
&\geq \cdots \\
&\geq \Pr\left[\texttt{Proper}_{[1]}\right] \left(1 - \frac{2}{2^{k-1}}\right)^{m-1} \\
&\geq \left(1 - \frac{2}{2^{k-1}}\right)^{m} > 0.
\end{aligned}$$

That is, the random coloring succeeds with positive probability in coloring the set X such that each A_i is properly colored. This concludes the proof that $\mathcal{F}$ is 2-colorable. □

The general formulation of the above theorem, talking about abstract "bad events" (instead of edges being monochromatic) in an abstract probability space (instead of the uniform space of two-colorings of X) is known as the Lovász Local Lemma. The Local Lemma was originally invented to properly color hypergraphs, but it went on to become one of the most fundamental tools of probabilistic combinatorics. The multiplicative factor in the above proof was optimized in Spencer [80]; its proof is an instructive exercise.

Theorem 9.2.3 (Lovász Local Lemma)**.** *Let $E_1, E_2, \ldots, E_m$ be events in some probability space. Let d and p be such that for every $i \in [m]$*

(1) *there exists a set* $\Gamma(i) \subseteq [m]$ *of at most* d *indices, such that* E_i *is mutually independent from the family* $\{E_j : j \in [m] \setminus (\Gamma(i) \cup \{i\})\}$ *of events, and*

(2) $\Pr[E_i] \le p$.

If $ep(d+1) < 1$, *then* $\Pr\left[\bigcap_{i=1}^m \overline{E_i}\right] > 0$.

Proof. Exercise. □

Let us practice the application of the Local Lemma on hypergraph 2-coloring.

Proposition 9.2.4. *Let* $\mathcal{F} \subseteq \binom{X}{k}$ *be a* k*-uniform hypergraph. Then*

(i) $\Delta(L(\mathcal{F})) \le \frac{2^{k-1}}{e} - 1 \;\Rightarrow\; \mathcal{F}$ *is 2-colorable.*

(ii) $\Delta(\mathcal{F}) \le \frac{2^{k-1}}{ek} \;\Rightarrow\; \mathcal{F}$ *is 2-colorable.*

Proof. Exercise. □

9.3 The Neighborhood Conjecture

The Local Lemma is a powerful existence statement, but unlike Claim 2.2.1 in Chapter 2, it is really just about existence: the guaranteed probability is exponentially small in the input size. Hence the following two problems arise naturally.

Problem 1. Can one find an efficient, ideally even deterministic, algorithm to properly two-color hypergraphs which satisfy the assumption of the Local Lemma?

Problem 2. Can one prove that if a hypergraph satisfies the conditions of the Local Lemma, then it is a Breaker's win?

In Chapter 2 the analogous two questions were discussed after proving Claim 2.2.1 with the first moment method. The Erdős-Selfridge Theorem answered both of them beautifully in the affirmative. In Chapter 2, however, the probability of success of the random coloring process was so high that potentially repeating it 100 times led to an efficient randomized algorithm. In the setup of the Local Lemma, on the other hand, the probability of success can be exponentially small, so even the possibility of an efficient randomized algorithm is unclear.

The success of the Erdős-Selfridge Criterion in making the first moment method simultaneously algorithmic **and** the basis of a game certainly provides an inspiration to study Problems 1 and 2 together. It might seem plausible that an affirmative answer to the algorithmic Problem 1 would come through positional games, maybe exactly via the positional game theoretic Problem 2.

Eventually this did not turn out to be the case, or at least not directly. While Problem 1 is completely solved by now, Problem 2 is still very much open. The ultimate solution of Problem 1 does not use positional games, but games were instrumental on the road to it. For Problem 1 the initial breakthrough was obtained by Beck [7] in 1990 using ideas he developed to study multidimensional Tic-Tac-Toe games. His efficient randomized algorithm 2-colored any k-uniform hypergraph $\mathcal{F}$ with $\Delta(L(\mathcal{F})) \leq 1.1^k$. Subsequently the base in the upper bound was improved several times by tweaking the original approach. A second breakthrough by Moser [66] was achieved in 2010, pushing the upper bound within a small constant factor of 2^k. Soon after, the problem was completely solved by Moser and Tardos [67]. They found an amazingly simple randomized algorithm, which properly 2-colors **any** hypergraph $\mathcal{F}$ satisfying the conditions of the Local Lemma in nearly linear time.

Concerning Problem 2, the following bold conjecture, labeled as a "game-theoretic Local Lemma", was formulated by József Beck.

Conjecture 9.3.1 (Neighborhood Conjecture, [10, Open Problem 9.1(a)])**.**

$$\Delta(L(\mathcal{F})) < 2^{k-1} - 1 \;\Rightarrow\; \mathcal{F} \textit{ is a Breaker's win.}$$

The particular upper bound in the conjecture on the maximum edge neighborhood size is even larger than the one from the Local Lemma (cf. Proposition 9.2.4), though other more sophisticated methods [76] ensure at least the existence of a proper 2-coloring in that large range (even up to $\Delta(L(\mathcal{F})) < c \cdot 2^k \sqrt{k/\ln k}$). The conjecture was clearly motivated by the construction of Erdős and Selfridge of a k-uniform Maker's win hypergraph $\mathcal{G}$ with 2^{k-1} edges, showing the tightness of their theorem (see Remark 2.3.6). The maximum neighborhood size in that construction is $2^{k-1} - 1$, as every pair of edges intersect. No better construction was known until 2009, when Gebauer [40] disproved the Neighborhood Conjecture by constructing Maker's win hypergraphs with maximum neighborhood size less than $0.74 \cdot 2^{k-1}$.

In his monograph [10], Beck also formulated several weakenings of his original conjecture. Maybe the most interesting form, which should probably inherit the name "Neighborhood Conjecture", is stated in terms of the maximum degree Δ instead of the maximum edge neighborhood size.

Conjecture 9.3.2 (The Reigning Neighborhood Conjecture, [10, Open Problem 9.1(d)])**.** *There is some $\varepsilon > 0$ such that*

$$\Delta(\mathcal{F}) < (1+\varepsilon)^k \;\Rightarrow\; \mathcal{F} \textit{ is a Breaker's win.}$$

The natural parameter to define for this conjecture is

$$D(k) := \min\{d : \exists\ k\text{-uniform Maker's win } \mathcal{F} \text{ with } \Delta(\mathcal{F}) \leq d\}.$$

The currently known best lower bound is $D(k) \geq \lfloor \frac{k}{2} \rfloor + 1$, which was an exercise in Chapter 1. This has been verified to be tight for $k = 3$ (folklore) and $k = 4$ by Knox [59]. The best known upper bound is outrageously far away, even $D(k) = \Omega\left(1.999^k\right)$ is a possibility. Deciding whether $D(k) = \lfloor \frac{k}{2} \rfloor + 1$ already seems to need new ideas.