Extremal Combinatorics Wi 2017–18
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Discrete Maths I Review

Do date: Oct 17th at 4:15 PM and Oct 18th at 8:30 AM.

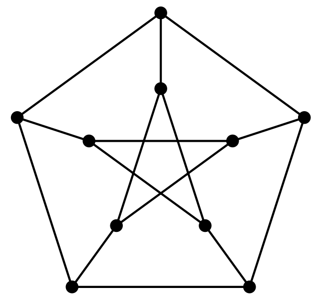
This sheet is intended to help you review some concepts from Discrete Mathematics I that might be useful for this semester's course. If you would like to review these topics together, come to the exercise class on the 17th of October or 18th of October, where you will get to discuss and work on these problems in small groups.

Exercise 1 Prove that on Facebook the number of people who have an odd number of friends is even.

Exercise 2 Show that for any set of S of n integers you can find a nonempty subset T of S with the property that the sum of all integers in T is divisible by n.

Exercise 3

(i) Determine the chromatic and independence numbers of the cycle C_n, and the following 10-vertex graph



(ii) Show that for any graph G, $\frac{v(G)}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$, where $\chi(G)$ and $\alpha(G)$ denote the chromatic and independence numbers respectively. Can you find examples where these bounds are tight?

Exercise 4 Let $k \in \mathbb{N}$ be a constant. For each pair of functions $f(n)$ and $g(n)$ from the functions given below, determine whether $f = o(g)$, $f = O(g)$, or $f = \Omega(g)$ (as n tends to infinity):

$$n^{\frac{1}{\log n}}, \binom{n}{k}, n^n, 2^{2^{\log^2 n}}, 2^{n^2}, n!, \log n.$$

Asymptotics and Estimates Let f, g be functions from $\mathbb{N}$ to $\mathbb{R}$. Then we use the following asymptotic notation:

- $f = o(g)$ if $\lim_{n\to\infty} f/g = 0$,
- $f = O(g)$ if there exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $|f(n)| \leq Cg(n)$ for all $n \geq n_0$, and
- $f = \Omega(g)$ if $g = O(f)$.
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Asymptotically, Stirling's approximation is the most powerful thing we have:

$$n! = (1 + O(1/n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

But we usually use the following estimates. Using $(1 + 1/k)^k \leq e$ for all $k \geq 1$, we can show that

$$\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

For binomial coefficients we have

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

The middle binomial coefficient can be estimated as follows.

$$\frac{2^{2k}}{2\sqrt{k}} \leq \binom{2k}{k} \leq \frac{2^{2k}}{\sqrt{2k}}.$$

Or, we can use Stirling's approximation to see the truth

$$\binom{2k}{k} = \frac{2^{2k}}{\sqrt{\pi k}}(1 + o(1))$$

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