Extremal Combinatorics Wi 2017–18
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Exercise Sheet 1

Due date: 16:15, 24th October

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In a `red`/`blue` colouring of the edges of K_n, let r_i denote the number of red edges with vertex i as an end point. Show that the number of monochromatic triangles is equal to $\binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$. Deduce that every `red`/`blue` colouring of $E(K_6)$ contains not only one, but at least *two* monochromatic triangles.

Exercise 2 In this exercise you will prove the classic Erdős–Szekeres bound on the Ramsey numbers. Recall that $R(s,t)$ is the minimum integer n for which every `red`/`blue`-colouring of the edges of K_n contains a monochromatic K_s in `red` or a monochromatic K_t in `blue`.

(i) Show that $R(s, 2) = s$ and $R(2, t) = t$.

(ii) Show that for any $s, t \geq 2$, $R(s, t) \leq R(s-1, t) + R(s, t-1)$.

(iii) Prove that $R(s, t) \leq \binom{s+t-2}{s-1}$ for any $s, t \geq 2$. (*Hint.* You might want to try induction on $s + t$.)

(iv) (Bonus) Conclude that for the symmetric Ramsey number we have $R(s) = O(\frac{4^s}{\sqrt{s}})$.

Remark. This gives us an improvement by a factor of $\sqrt{s}$, over what we have proved in the lecture—which pales unfortunately, when compared to the exponential factor gap between the upper and lower bounds we know.

Exercise 3

(a) Show that $R(3, 4) \leq 10$.

(b) Improve (a) to $R(3, 4) \leq 9$.

(c) Show that $R(3, 4) = 9$.

Exercise 4 For the multicolour Ramsey numbers prove that

$$R_r(t_1, t_2, \ldots, t_r) \leq r^{1+\sum_{i=1}^{r}(t_i-1)}.$$