Extremal Combinatorics — Wi 2017–18

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Exercise Sheet 10

Due date: 16:15, 23rd January

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 (Two-distance sets) A set S of points in $\mathbb{R}^n$ is called a **two-distance set** if the distance between pairwise distinct points of S takes only two distinct values. Let $m(n)$ be the maximum size of a two-distance set in $\mathbb{R}^n$.

(1) Prove that $m(n) \geq \binom{n}{2}$.

(2) By constructing an appropriate linearly independent set of polynomials, show that $m(n) \leq (n+1)(n+4)/2$.

(3) (Bonus) Improve the upper bound on $m(n)$ to $\binom{n+2}{2}$ and prove that $m(2) = 5$.

Exercise 2 Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of subsets of $[n]$ for which we have $|F_i \cap F_j| \equiv 0 \pmod 2$ for all $i \neq j$.

(1) Prove that

$$m \leq k + 2^{\lfloor \frac{n-k}{2} \rfloor},$$

where k is the number of elements in $\mathcal{F}$ which have odd cardinality.

(2) Deduce that

$$m \leq \begin{cases} n+1 & \text{if } n \leq 5 \\ 2^{n/2} & \text{if } n \text{ is even and } n \geq 6 \\ 1 + 2^{(n-1)/2} & \text{if } n \text{ is odd and } n \geq 7 \end{cases}$$

Exercise 3 Let UD_n be the unit distance graph in $\mathbb{R}^n$. Prove that $\chi(\mathrm{UD}_n) \leq 9^n$ for all n.

Exercise 4 Let X be a finite set and $\mathbb{F}$ a field. Recall that the tensor rank of a function $M : X \times X \times X \mapsto \mathbb{F}$ is the least n for which there exists a decomposition of M as $M(x, y, z) = \sum_{i=1}^{n} f_i(x)g_i(y)h_i(z)$, while the slice rank of M is the least r for which there exists a decomposition of M as $M(x, y, z) = \sum_{i=1}^{k} f_i(x)g_i(y, z) + \sum_{i=k+1}^{m} f_i(y)g_i(x, z) + \sum_{i=m+1}^{n} f_i(z)g_i(x, y)$ where $0 \leq k \leq m \leq n$.

(1) Prove that the tensor rank of a function is at most $|X|$ times its slice rank.

(2) Give an example of a function $M : X \times X \times X \mapsto \mathbb{F}$ for which the slice rank of M is equal to 1 and the tensor rank of M is equal to $|X|$.

Exercise 5 (Bonus) Let $q = p^r$ be a prime power, with r a positive integer and p a prime. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a set family in $2^{[n]}$ with the property that $|F_i| \equiv k \pmod q$ and $|F_i \cap F_j| \not\equiv k \pmod q$ for all $i \neq j$, for some integer k.

(1) Prove that for any positive integer x, we have

$$\binom{x-1}{q-1} \equiv \begin{cases} 1 \pmod p & \text{if } x \equiv 0 \pmod q \\ 0 \pmod p & \text{if } x \not\equiv 0 \pmod q \end{cases}.$$

(2) Use the function from (1) (and the ideas from the lectures) to prove that $m \leq \binom{n}{q-1}$.

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Exercise 3: As a first step, construct a maximal set $S \subseteq \mathbb{R}^n$ of points such that the distance between any two of them is at least $1/2$. Properly colour the infinite graph formed on the point set S by making two points adjacent if their distance is at most 2, using at most 9^n colours, and then use this colouring to obtain a proper colouring of the unit distance graph.
Exercise 5: Use Lucas's theorem (on binomial coefficients modulo a prime).