Extremal Combinatorics Wi 2017–18
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 11

Due date: 16:15, 30th January

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Show that the skew version of the Bollobás set-pairs inequality is false in the non-uniform setting. More specifically, for each $n \in \mathbb{N}$, find a sequence of sets $A_1, \dots, A_m$ and $B_1, \dots, B_m$ such that

(1) $A_i \cap B_i = \emptyset$ for all i, and

(2) $A_i \cap B_j \neq \emptyset$ for all $i > j$, but

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \geq n+1.$$

Exercise 2 Let $\mathbb{F}_p$ be the finite field of order p, where p is a prime number [1]. Recall that a set of vectors S in the n-dimensional vector space $\mathbb{F}_p^n$ is said to be in *general linear position* if every n element subset of S forms a linearly independent set of vectors. Construct a set of $\max\{p, n\} + 1$ vectors in $\mathbb{F}_p^n$ which are in general linear position.

Exercise 3 For a graph H let $\mathrm{sat}(n, H)$ and $\mathrm{wsat}(n, H)$ denote the saturation number and the weak saturation number.

(1) Prove that $\mathrm{sat}(7, C_4) > 7$ and $\mathrm{wsat}(7, C_4) \leq 7$.

(2) (Bonus) Prove that $\mathrm{sat}(n, C_4) \leq \lfloor (3n-5)/2 \rfloor$ and $\mathrm{wsat}(n, 4) \leq n$ for all $n \geq 5$.

Remark It is in fact known that $\mathrm{sat}(n, C_4) = \lfloor (3n-5)/2 \rfloor$ and $\mathrm{wsat}(n, C_4) = n$ for all $n \geq 5$, but proving the lower bounds is quite difficult. This shows that the saturation number and the weak saturation number can be very different (and difficult to compute), even for a graph like C_4.

[1] That is, the set of residues of integers modulo p.

Exercise 4 Let $\mathcal{F}$ be a k-uniform set family which is intersection free, i.e., for any three distinct members of A, B, C of $\mathcal{F}$, we have $A \cap B \not\subseteq C$. Then show that

$$|\mathcal{F}| \leq 1 + \binom{k}{\lfloor k/2 \rfloor}.$$

Exercise 5 (Bonus)

(1) Prove that the number of k-dimensional vector subspaces of $\mathbb{F}_p^n$ is equal to

$$\frac{(p^n - 1)(p^n - p) \cdots (p^n - p^{k-1})}{(p^k - 1)(p^k - p) \cdots (p^k - p^{k-1})}.$$

(2) Prove that for any linearly independent set C of $n - 2$ vectors in $\mathbb{F}_p^n$, there exist exactly $p + 1$ subspaces V of $\mathbb{F}_p^n$ which contain C and have dimension $n - 1$.

(3) Use (2) to deduce that any set of vectors in $\mathbb{F}_p^n$ in general linear position has cardinality at most $p + n - 1$.

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Exercise 1: you can do this with a sequence of 2^n pairs of subsets of $[n]$.