Extremal Combinatorics Wi 2017–18
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Exercise Sheet 12

Due date: 16:15, 6th February

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Given two natural numbers k and n, the k-cascade representation of n is given by

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

where $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$. Prove that such a representation of n exists and is unique.

Exercise 2 The **colexicographical order** on k-susbets of $\mathbb{N}$ is defined as $A < B$ if and only if $\max(A \triangle B) \in B$.

(1) Prove that this gives a total order, and that for any positive integer n the first $\binom{n}{k}$ elements with respect to this order are precisely the k-subsets of $\{1, \ldots, n\}$ (in colexicographical order).

(2) Let m be a positive integer and let $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ be its k-cascade representation. Prove that the family of the first m elements of $\binom{\mathbb{N}}{k}$ with respect to the colexicographical order have a shadow of size $\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}$.

Exercise 3 Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$, define its ℓ-shadow to be

$$\partial_\ell(\mathcal{F}) = \{E \in \binom{[n]}{\ell} : E \subset F \text{ for some } F \in \mathcal{F}\}.$$

(1) For $0 \leq \ell < k$ and $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ for $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$, determine the smallest possible size of the ℓ-shadow of a set family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size m.

(2) Deduce the Erdős-Ko-Rado theorem: if $n \geq 2k$, then the largest intersecting family in $\binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.

Exercise 4 In this exercise we will prove a slightly stronger version of the 2-dimensional case of Sperner's Lemma.

Let S be a triangle in $\mathbb{R}^2$, and suppose its vertices are coloured $1, 2$ and 3 in clockwise order. Now consider any subdivision of S with a legal Sperner colouring of the new vertices. For each subtriangle in the subdivision, we will assign a label to each of its three edges. Processing the colours of the vertices in clockwise order, give an edge label $+1$ if the colours of its endpoints are $(1, 2), (2, 3)$ or $(3, 1)$. Give it label -1 if the colours are $(2, 1)$, $(3, 2)$ or $(1, 3)$. Give it label 0 if the colours are $(1, 1)$, $(2, 2)$ or $(3, 3)$. Note that as an internal edge of the subdivision is contained in two subtriangles, it will receive two labels: one for each subtriangle it is in.

By considering the sums of all of the labels, show that the subdivision does not just contain a rainbow triangle, but it contains a rainbow triangle with vertices of colour 1, 2 and 3 in clockwise order.

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Exercise 3: Find a way to convert disjoint sets into comparable sets, giving a link between intersecting families and shadows of set families.