Extremal Combinatorics Wi 2017–18
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Exercise Sheet 13

Due date: 16:15, 13th February

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise we will prove that the only extremal set families with respect to Sperner's theorem [1] are the "mid-levels", using Kruskal-Katona theorem.

While Kruskal-Katona gives a pretty sharp bound on the shadow size, in practice it is usually better to use the follow weaker, but computationally friendlier, version due to Lovász. For $x \in \mathbb{R}$, and $k \in \mathbb{N}$, define

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

for all $x \geq k$. Since this is an increasing (continuous) function in this domain, we see that for every integer m there exists a unique x for which $m = \binom{x}{k}$.

Theorem 1 (Lovász). *Let $\mathcal{F}$ be a k-uniform set-family of size $m = \binom{x}{k}$, for some real number $x \geq k$. Then*

$$|\partial F| \geq \binom{x}{k-1}.$$

Let $n = 2k+1$ be an odd integer, and let $\mathcal{F}$ be an antichain in $2^{[n]}$ of largest possible size. Prove that $\mathcal{F}$ is either equal to $\binom{[n]}{k}$ or $\binom{[n]}{k+1}$ (and so it has to consist of all elements of one of the mid-level layers).

Exercise 2 Let B^d denote the closed unit ball in $\mathbb{R}^d$; that is,

$$B^d = \left\{ \vec{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 \leq 1 \right\}.$$

Let $f : B^d \to B^d$ be a continuous function. Brouwer's Fixed Point Theorem states that f must have a *fixed point*; that is, there is some $\vec{x} \in B^d$ such that $f(\vec{x}) = \vec{x}$.

[1] not the Sperner's lemma!

Since B^d can be continuously and reversibly deformed into the standard d-simplex Δ^d, where

$$\Delta^d = \left\{ \vec{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 1, \text{ and } x_i \geq 0 \text{ for all } i \right\},$$

it is equivalent to show that any continuous map $f : \Delta^d \to \Delta^d$ has a fixed point. Use Sperner's Lemma to prove this version of Brouwer's Fixed Point Theorem.

Exercise 3 Let G be an $(2k)$-partite graph, with each part having n vertices, of maximum degree Δ.

(i) Show that if $n > 2\Delta - \frac{\Delta}{k}$, then G must have an independent transversal.

(ii) Show that this is best possible: construct a $(2k)$-partite graph with parts of size $2\Delta - \lceil \frac{\Delta}{k} \rceil$ and maximum degree Δ that has no independent transversals.

Exercise 4 We define the *Kneser graph* $KG(n, k)$ to have vertices $V = \binom{[n]}{k}$, with edges $F_1 \sim F_2$ if and only if $F_1 \cap F_2 = \emptyset$. Observe that $KG(5, 2)$ is the well-known Petersen graph. Given a graph G, let (G) be the set of its independent sets. The *fractional chromatic number* $\chi_f(G)$ is defined as the minimum $r \in \mathbb{R}$ for which one may assign non-negative real numbers $x_I \geq 0$ to every independent set $I \in (G)$ such that $\sum_{I \in (G)} x_I = r$, subject to the constraint that for every vertex $v \in V(G)$, $\sum_{I \ni v} x_I \geq 1$.

(ii) Show that for any N-vertex graph G, $\frac{N}{\alpha(G)} \leq \chi_f(G) \leq \chi(G)$.

(iii) When $n \geq 2k$, show that $\chi_f(KG(n, k)) = \frac{n}{k}$.

HINTS

Exercise 1: Use the fact that the shadow of the intersection of the antichain with the $(k + 1)$st-level is disjoint from the intersection of the antichain with the k-th level.

Exercise 2: Consider a subdivision with simplices of smaller diameter. Colour each vertex x of the subdivision based on which coordinate of x is larger than in its image under f. Apply Sperner's Lemma, and take a limit.