Extremal Combinatorics Wi 2017–18
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Exercise Sheet 2

Due date: 16:15, 7th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Deduce the finite Ramsey theorem from the infinite case. That is, using the fact that every `red`/`blue` colouring of $\binom{\mathbb{N}}{2}$ contains an infinite monochromatic clique, show that for every positive integer $t \geq 2$, there exists a finite n such that every `red`/`blue` colouring of $E(K_n)$ contains a monochromatic K_t.

Exercise 2 Recall the definition of the multicolour Ramsey number $R_r(t_1, \dots, t_r)$ and the upper bound that we proved in the previous exercise sheet: $R_r(t_1, \dots, t_r) \leq r^{1+\sum(t_i-2)}$. [1] When $t_1 = \cdots = t_r = 3$, this gives us $R_r(3, \dots, 3) \leq r^{r+1}$. In this exercise we will improve this upper bound.

(a) Prove that $R_r(3, \dots, 3) \leq 2 + r(R_{r-1}(3, \dots, 3) - 1)$, for all $r \geq 3$.

(b) From (a), deduce that $R_r(3, \dots, 3) \leq 1 + \lfloor er! \rfloor$. (Hint: $e = \sum_{i=0}^{\infty} \frac{1}{i!}$).

Exercise 3 We have already seen that $R_r(t, \dots, t) \leq r^{r(t-2)+1}$. We now look at lower bounds on this multicolour Ramsey number.

(a) Show that if $r\binom{n}{t}r^{-\binom{t}{2}} < 1$, then $R_r(t, t, \dots, t) \geq n + 1$. Deduce the bound

$$R_r(t, t, \dots, t) \geq te^{-1}r^{\frac{t-1}{2}-\frac{1}{t}}.$$

(b) Now fix $t = 3$. Then the lower bound that we have just proved gives us $R_r(3, \dots, 3) \geq 3e^{-1}r^{2/3}$, which is pretty bad compared to the upper bound of $O(r!)$ that we obtained in **Exercise 2**. Prove by an explicit construction that $R_r(3, \dots, 3) \geq 2^r$.

Remark: This shows how things can be dramatically different when we move from the regime of fixed number of colourings but growing size of the monochromatic cliques to the regime of fixed size of the monochromatic clique but growing number of colours.

[1] We proved $R_r(t_1, \dots, t_r) \leq r^{1+\sum(t_i-1)}$ but in fact this sharper estimate also follows from the same proof(s).

Exercise 4 Show that if $n \geq R^{(3)}(t,t)$, then any set of n points in $\mathbb{R}^2$, no three collinear, contains a subset of size t which forms a convex set. (Hint: t points in general position form a convex set if and only if every 4 element subset forms a convex set; now describe 4 element convex sets in terms of their 3-element subsets.)