Extremal Combinatorics Wi 2017–18
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Exercise Sheet 3

Due date: 16:15, 14th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let r, k be integers with $r \geq 2$ and $k \geq 1$. Prove that for any r-colouring of $\binom{\mathbb{N}}{k}$, there exists an infinite subset S of $\mathbb{N}$ such that every element of $\binom{S}{k}$ receives the same colour.

Exercise 2 By considering a random colouring, show that

$$R^{(k)}(t,t) \geq (1-o(1))(t/e)2^{\frac{1}{k}\binom{t-1}{k-1}} \geq (1-o(1))2^{c_k t^{k-1}},$$

where c_k is some constant that only depends on k.

Remark: From this we see that while the random colouring gave good lower bounds for the Ramsey number $R(t,t) = R^{(2)}(t,t)$, the lower bound that we get for hypergraph Ramsey numbers are really far away from the tower-type upper bounds. In fact, we know much better lower bounds for these numbers than what we have proved here.

Exercise 3 Let H be a k-graph, for some $k \geq 2$, with the property that $|e \cap f| \neq 1$ for any two edges $e, f \in E(H)$. Show that H is two-colourable.

Exercise 4 In this exercise we will prove an upper bound on the happy ending number $\mathrm{HE}(t)$ without relying on the hypergraph Ramsey numbers. Recall that $\mathrm{HE}(t)$ is the smallest number of points in $\mathbb{R}^2$ in general position which ensure that we always have t points from them, forming a convex subset.

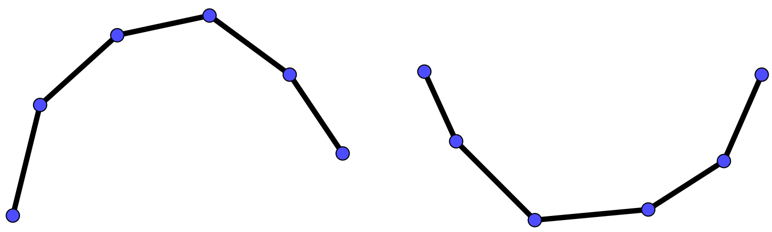
A sequence of consecutive line segments in $\mathbb{R}^2$ is called a **Cap** if their slopes are monotonically decreasing, and a **Cup** if their slopes are monotonically increasing (see the figure below). Let $f(s,t)$ denote the smallest number for which any collection of $f(s,t)$ points in general position either contains a **Cap** of length s or a **Cup** of length t, where the length of a **Cap/Cup** is equal to the number of points contained in it.

(a) Prove that $f(s,3) = s$ and $f(3,t) = t$ for all $s, t \geq 3$.

(b) Prove that $f(s,t) \le f(s-1,t) + f(s,t-1) - 1$ for all $s,t \ge 4$. (Hint: what if there are at least $f(s,t-1)$ points that are the left most points of the **Caps** of length $s-1$?)

(c) Deduce that
$$f(s,t) \le \binom{s+t-4}{s-2} + 1.$$

(d) Show that the happy ending number can be bounded from above as follows:
$$\mathrm{HE}(t) \le \binom{2t-4}{t-2} + 1.$$



Cap **Cup**

Remark: The bound that we have obtained is much better than any of the upper bounds obtained using the hypergraph Ramsey numbers. Indeed, the upper bound we gave in the lecture using four-uniform Ramsey numbers gives a tower of height four, the one you proved in the last homework using the three-uniform Ramsey numbers gives a tower of height three and our bound here is the single exponential 2^{2t}. The lower bound given by Erdős and Szekeres was $2^{t-2}+1$ and this is what they conjectured to be the truth. After essentially no significant progress for the last 80 years, the upper bound was recently improved to $2^{t(1+o(1))}$ by Andrew Suk [1].

[1] http://www.ams.org/journals/jams/2017-30-04/S0894-0347-2016-00869-X/home.html