Extremal Combinatorics Wi 2017–18
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 4

Due date: 16:15, 21st November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Recall that $m_B(k)$ is the smallest number m for which there exists a k-graph on m hyperedges which is non-two-colorable.

(1) Prove that the following 3-uniform hypergraph on 7 edges is non-two-colorable. Deduce that $m_B(3) \leq 7$.

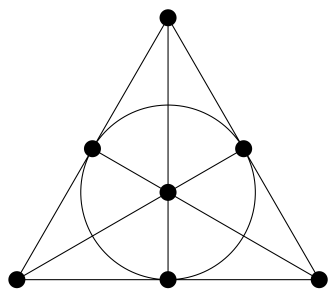

Figure 1: The Fano plane

(2) Prove that $m_B(3) = 7$.

Exercise 2 In this exercise, you will complete the details of the Cherkashin–Kozik lower bound on the size of the smallest non-two-colourable k-graph. Let H be a k-graph with m edges. Recall that for each vertex $v \in V(H)$, we independently sample $x_v \sim U([0,1])$, a uniformly random number in $[0,1]$. We then order the vertices in increasing order of x_v, and run the greedy algorithm in that order. That is, we colour a vertex blue unless it is the last vertex in an all-blue edge, in which case it is coloured red.

(a) Consider the following events, where $\delta \in (0,1)$.

(i) $\mathcal{L}_e = \left\{ \forall v \in e : x_v < \frac{1}{2}(1-\delta) \right\}$ for some edge $e \in E(H)$.

(ii) $\mathcal{R}_f = \left\{\forall v \in f : x_v > \frac{1}{2}(1+\delta)\right\}$ for some edge $f \in E(H)$.

(iii) $\mathcal{E}_{e,f} = \{|e \cap f| = 1,$ the last vertex v of e is the first vertex of f, and $x_v \in \left[\frac{1}{2}(1-\delta), \frac{1}{2}(1+\delta)\right]\}$ for two edges $e, f \in E(H)$.

Show that $\mathbb{P}(\mathcal{L}_e) = \mathbb{P}(\mathcal{R}_f) = (1-\delta)^k 2^{-k}$ and $\mathbb{P}(\mathcal{E}_{e,f}) \leq \delta 2^{2-2k}$.

(b) Let $m = \beta 2^{k-1}$. Show that if $\beta(1-\delta)^k + \beta^2\delta < 1$, then H is two-colourable.

(c) By choosing β and δ appropriately, show that there is some positive constant $c > 0$ such that $m_B(k) \geq c\left(\frac{k}{\ln k}\right)^{\frac{1}{2}} 2^k$.

Exercise 3 Recall that the Turán number $\mathrm{ex}(n, H)$ for a graph H is the largest integer e such there is a graph on n vertices and e edges which does not contain any copy of H. Determine the exact value of $\mathrm{ex}(n, K_{1,4})$ and $\mathrm{ex}(n, P_4)$, where P_4 is the path on 4 vertices.

Exercise 4 In this exercise you will determine the asymptotic behaviour of $\mathrm{ex}(n, C_4)$. Let $\mathbb{F}_q$ be the finite field of order q, where q is a prime power [1]. Let $\mathcal{P} = \{(x, y) : x, y \in \mathbb{F}_q\}$ be the set of points in the plane $\mathbb{F}_q^2$ and let

$$\mathcal{L} = \{\{(x, y) : ax + by = c\} : a, b, c \in \mathbb{F}_q \text{ and at least one of } a, b \text{ is non-zero}\}$$

be the set of lines. Define a bipartite graph G_q with the two parts as $\mathcal{P}$ and $\mathcal{L}$ and making an edge between a point $P = (x, y)$ and a line ℓ if P lies on ℓ, i.e., its coordinates (x, y) satisfy the equation of ℓ.

(1) Show that the graph G_q has $2q^2 + q$ vertices and $q^3 + q^2$ edges.

(2) Show that the graph G_q does not contain any C_4's, and thus deduce that there exists a constant C such that for infinitely many n, we have $\mathrm{ex}(n, C_4) \geq Cn^{3/2}$.

(3) Conclude that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2}))$ (Hint: Chebyshev's theorem [2]).

Bonus The infinite hypergraph Ramsey theorem shows that, for any $k \geq 1$, if you r-colour all k-sets of natural numbers, $\binom{\mathbb{N}}{k}$, you can find an infinite set $S \subset \mathbb{N}$ such that $\binom{S}{k}$ is monochromatic. Moreover, even if we use infinitely many colours, we have a canonical Ramsey theorem which says that there must be an "orderly" structure (as discussed in the lectures).

What happens if we instead colour the *infinite* subsets of $\mathbb{N}$? Given an infinite set S, let $\binom{S}{\omega}$ denote the set of all infinite subsets of S. If we `red`/`blue`-colour $\binom{\mathbb{N}}{\omega}$, must we find an infinite set $S \subset \mathbb{N}$ such that $\binom{S}{\omega}$ is monochromatic? (Hint: use axiom of choice)

[1] for example, when q is equal to a prime p you can take this to be the residues modulo p

[2] "Chebyshev said it, and I say it again,
there is always a prime between n and $2n$." (bonus: find out who said this!)