Extremal Combinatorics Wi 2017–18

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Exercise Sheet 5

Due date: 16:15, 28th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1

(a) Show that every graph G of average degree d contains a subgraph with minimum degree greater than $d/2$.

(b) Show that for any tree T on t vertices $\mathrm{ex}(n,T) \leq (t-2)n$.

(c) Show that for any tree T with t edges, $\mathrm{ex}(n,T) \geq \frac{(t-2)n}{2}$ for infinitely many n.

(d) Show that in the special case of the star graph $T = K_{1,t-1}$ the lower bound is tight.

Remark. Part (b) and (c) essentially determines the Turán number of every tree T on t vertices up to a constant factor 2. Part (d) gives an example of a tree where the lower bound is tight. The Erdős-Sós Conjecture (1962) states that the Turán number of *every* tree on t vertices should be equal to the lower bound. The conjecture is known to hold for paths and many other trees. Ajtai, Komlós, Simonovits, and Szemerédi announced the proof of the full conjecture but the manuscript have not appeared yet.

Exercise 2 In this exercise you will prove the Kővari-Sós-Turán theorem. Let G be an n-vertex $K_{s,t}$-free graph.

(1) Show that the number of copies of $K_{1,s}$ in G is equal to

$$\sum_{v\in V(G)} \binom{d(v)}{s}.$$

(2) Show that the number of copies of $K_{1,s}$ is at most $(t-1)\binom{n}{s}$.

(3) Deduce that $\mathrm{ex}(n, K_{s,t}) \leq c_t n^{2-1/s}$, for some constant c_t that depends on t.

Exercise 3 The TV remote of George requires two working batteries to function. Opening the drawer in which he keeps the batteries, he finds eight. He remembers that four of them work and four of them do not, but there is no way of telling them apart without testing them in the remote. How quickly can George guarantee to find two working batteries for his remote? (i.e. what is the minimum number of tests that will suffice even in the worst case?)

Exercise 4 Let A be a set of n points in $\mathbb{R}^2$. Prove that the number of unit distances determined by A, i.e., the number of pairs $\{x, y\} \subset A$ with $\mathrm{d}(x, y) = 1$, is at most $O(n^{3/2})$.

Practice exercise: asymptotic notation Let $T_{n,r}$ be the complete r-partite graph on n vertices in which the size of any two parts differs by at most 1. Show carefully that

$$|E(T_{n,r})| = \left(1 - \frac{1}{r}\right)\binom{n}{2} + O(n),$$

assuming r to be a constant.

HINTS

Exercise 1 (a): ytpmenon si revotfel eht taht evorp dna $2\backslash d$ tsom ta eerged htiw secitrev eteled ylevitaretI

Exercise 2: $1 - s \le x$ rof si ti tub ,x fo seulav laer fo egnar lluf eht ni noitcnuf xevnoc a ton si $\binom{x}{s}$,$2 < s$ rof taht etoN

Exercise 4: secnatsid tinu fo rebmun eht serutpac hcihw fo rebmun-egde eht hparg yrailixua etairporppa na enifeD