Extremal Combinatorics Wi 2017–18
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Exercise Sheet 6

Due date: 16:15, 5th December

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let G be a graph on n vertices which has no cycles of length less than g.

(a) If the degree of each vertex in G is at least d, then prove that the number of vertices in G is $\Omega(d^k)$ where $k = \lfloor \frac{g-1}{2} \rfloor$.

(b) Prove that the number of edges in G is at most $O(n^{1+1/k})$ where $k = \lfloor \frac{g-1}{2} \rfloor$.

Exercise 2 Suppose G is a graph with $\chi(G) = k$, and let $V = V_1 \cup V_2 \cup \ldots \cup V_k$ be a proper k-colouring of G (where each vertex in V_i receives colour i).

(a) Show that for every $1 \leq i \leq k$, there is a vertex $v_i \in V_i$ such that v_i has a neighbour in V_j for every $j \neq i$.

(b) Deduce that G has at least $\binom{k}{2}$ edges.

(c) If G is also triangle-free, improve this lower bound to $e(G) \geq \frac{3}{4}k^2 - k$.

Remark: Part (b) shows that the smallest number of edges in a non $(k-1)$-colorable graph is $\binom{k}{2}$. Compare this to the Property B of hypergraphs.

Exercise 3 Let $G = (V, E)$ be a graph, and let $A, B \subset V$ be two disjoint non-empty sets of vertices, of sizes a and b respectively. Let $X \subset A$ be a uniformly random set of k vertices, where $1 \leq k \leq a$ is fixed, and let $Y \subset B$ be an independent uniformly random set of ℓ vertices, for some fixed $1 \leq \ell \leq b$.

(a) Show that $\mathbb{E}[d(X, Y)] = d(A, B)$; that is, the average density over subsets is the density of the pair (A, B) itself.

(b) Deduce that, in order to check whether or not the pair (A, B) is ε-regular, it suffices to check if $|d(X, Y) - d(A, B)| \leq \varepsilon$ for every pair (X, Y) with $X \subset A$ of size precisely $\lceil \varepsilon a \rceil$ and $Y \subset B$ of size precisely $\lceil \varepsilon b \rceil$.

Exercise 4 In this exercise you will show that a random bipartite graph is ε-regular with high probability, thus validating our intuitive understanding that ε-regular pairs are "random looking".

Fix some $\varepsilon > 0$ and some probability $p \in [0, 1]$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the two parts of the random bipartite graph $G(n, n, p)$, i.e. for all $1 \leq i, j \leq n$, the edge $\{a_i, b_j\}$ belongs to our random graph G with probability p, independently of all other edges. Show that

$$\mathbb{P}\left((A, B) \text{ is } \varepsilon\text{-regular}\right) \to 1 \quad \text{as} \quad n \to \infty.$$

You may use the Chernoff bound, which in particular implies that if a random variable $X \sim \mathrm{Bin}(N, p)$ is binomially distributed with parameters

$$\mathbb{P}\left(|X - Np| \geq t\right) \leq 2e^{-2t^2/N}.$$

HINTS

Exercise 1(b): .eerged egareva egral fo shparg ni eerged muminim hgih fo hpargbus afo ecnetsixe ehttuoba krowemoh suoiverp eht ni devorp uoy ammel eht esU