Extremal Combinatorics Wi 2017–18
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Exercise Sheet 7

Due date: 16:15, 12th December

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 (Property Testing.)

(a) Prove that for every $\varepsilon > 0$, there exists an $f(\epsilon) > 0$ such that for every graph G on vertex set $[n]$, there exists a randomized algorithm that queries at most $f(\varepsilon)$ pairs of $[n]$ (whether they are an edge of G) and outputs either `accept` or `reject`, such that

- if G is triangle free, then the algorithm outputs `accept` with probability 1,
- if G is ϵ-far from triangle free, then G outputs `reject` with probability 0.9999.

(b) Let `Alg-Triangle` be an algorithm that queries pairs of $[n]$ (whether they are edges of some (hidden) input graph G on vertex set $[n]$) and determines (with 100% certainty) whether G is triangle-free or not. Show that in the worst case `Alg-Triangle` needs to query at least $(\frac{1}{4} - o(1))n^2$ pairs.

Exercise 2 (Common Neighborhood Lemma.) Let (A, B) be an ε-regular pair of density at least d. Let $s \in \mathbb{N}$ be fixed integer and $Y \subseteq B$ a fixed subset of vertices, for which $(d-\varepsilon)^{s-1}|Y| \geq \varepsilon |B|$. Then

$$|\{(a_1, a_2, \ldots, a_s) \in A^s : |Y \cap (\cap_{i=1}^{s} N(a_i))| < (d-\varepsilon)^s |Y|\}| \leq s\varepsilon |A|^s,$$

where $N(a)$ is the neighbourhood of a vertex a.

Exercise 3 The half-graph H_n is a bipartite graph with parts $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and edges $E = \{\{a_i, b_j\} : i \leq j\}$. Given $\varepsilon > 0$, and any even $k \in \mathbb{N}$ such that $k \geq 1 + 1/\varepsilon$, find an explicit ε-regular partition of H_n with k parts (not counting the exceptional set) when $n \geq k/\varepsilon$. How many irregular pairs does your partition have?

Bonus: Show that for every $\varepsilon > 0$ there is some constant $c = c(\varepsilon) > 0$ such that if $k \geq 2$ and n is large enough, any equipartition of H_n into k parts must have at least ck irregular pairs. [1]

[1] This shows that one cannot remove irregular pairs from the statement of the regularity lemma. Interestingly, it has been shown that these half-graphs are "essentially" the only obstruction to removing these irregular pairs: `http://www.ams.org/journals/tran/2014-366-03/S0002-9947-2013-05820-5/home.html`

Exercise 4 Show that for every $\gamma > 0$ there exists a $\delta(\gamma) > 0$, such that if G is an n-vertex graph with fewer than δn^4 copies of K_4, then G can be made K_4-free by removing at most γn^2 edges.