Extremal Combinatorics Wi 2017–18
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Exercise Sheet 8

Due date: 16:15, 19th December

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 (Schur's Theorem)

(1) Show that whenever the natural numbers $\mathbb{N}$ are finitely colored, one can find monochromatic $x, y, z \in \mathbb{N}$ such that $x + y = z$.

(2) Prove that for every positive integer r, there exists a positive integer $S(r)$ such that every r-coloring of $[S(r)]$ contains a monochromatic triple x, y, z with $x + y = z$.

(3) (Bonus) Fix a positive integer n. Prove that for every prime $p \geq S(n) + 1$, the equation $x^n + y^n = z^n$ has a solution modulo p.

Remark: The third part is somewhat related to Fermat's last theorem. It shows that for any given n the equation $x^n + y^n = z^n$ has solutions modulo p for all but finitely many primes p. In fact this results was discovered by Schur in an attempt to prove Fermat's last theorem (and perhaps to present a new proof of Dickson's theorem [1]), which says that there are no solutions of this equation over the integers, for $n > 2$.

Exercise 2 Prove that the set of positive integers can be 4-colored such that no color class contains a solution of $x + y = 3z$.

Exercise 3 Prove, or disprove, the following variation of Van der Waerden's theorem: for every two-coloring of the natural numbers, there exists an *infinite* monochromatic arithmetic progression.

[1] See "J. Nešetřil and M. Rosenfeld. I. Schur, C. E. Shannon and Ramsey numbers, a short story. Discrete Math., 229:185–195, 2001" for more on this

Exercise 4 Show that if we construct a subset R of non-negative integers in the following greedy fashion: consider the integers in increasing order and place the next integer into R if this does not create a 3-AP with the elements that are already in R, then R is equal to the set of all non-negative integers which do not have the digit 2 in their ternary representation.

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Exercise 2: Every natural number n can be written as $n = 5^{a_n}(5b_n + c_n)$ where $1 \leq c_n \leq 4$.

Exercise 1 (3): for every prime p, there exists a non-zero element θ in $\mathbb{Z}/p\mathbb{Z}$ such that every non-zero element of $\mathbb{Z}/p\mathbb{Z}$ can be written as a power of θ.