Extremal Combinatorics — Wi 2017–18

Tibor Szabó
Anurag Bishnoi

Exercise Sheet 9

Due date: 16:15, 16th January

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 For two vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field, let $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$ be the standard inner product [1]. For a set $S \subseteq \mathbb{F}^n$, let $S^\perp := \{\mathbf{v} \in \mathbb{F}^n : \mathbf{v} \cdot \mathbf{u} = 0 \ \forall \mathbf{u} \in S\}$.

(1) Prove that if U is a (vector) subspace of $\mathbb{F}^n$, then $U^\perp$ is also a subspace and $\dim U + \dim U^\perp = n$.

Recall that in an Eventown we have distinct subsets of $[n]$ (clubs from a town of n people) such that every subset has even cardinality and every two distinct subsets share an even number of elements.

(2) Prove that in an *Eventown* the number of subsets is at most $2^{\lfloor n/2 \rfloor}$.

Exercise 2 (Equiangular Lines) A set S of lines through origin in the n-dimensional Euclidean space $\mathbb{R}^n$ is called **equiangular** if every pair of lines in S has the same angle between them. In this exercise we will prove an upper bound on the number of equiangular lines in $\mathbb{R}^n$ using the linear algebra method.

(1) Give an example of such a set S in $\mathbb{R}^2$ with $|S| = 3$. Can we construct a larger equiangular set S in $\mathbb{R}^2$?

(2) Let $S = \{L_1, \ldots, L_k\}$ be a set of equiangular lines in $\mathbb{R}^n$ and let $\alpha \neq 1$ be the cosine of the common angle between these lines. For each i, let $\mathbf{u}^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})$ be an arbitrary unit vector on L_i. Use these $\mathbf{u}^{(i)}$'s to construct polynomial functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in \{1, \ldots, k\}$, for which we have

$$f_j(\mathbf{u}^{(i)}) = \begin{cases} 1 - \alpha^2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

and show that $f_1, \ldots, f_k$ are linearly independnet. Choose f_i's appropriately and deduce that $|S| \leq n(n+1)/2 + 1$.

[1] it's more standard to call this a symmetric bilinear form and reserve the term "inner product" for the case when $\mathbb{F}$ is equal to $\mathbb{R}$ or $\mathbb{C}$.

(3) Improve the upper bound on the number of equiangular lines to $n(n+1)/2$.

Remark: This bound was proved by Gerzon in early 70's and the proof above was given by Koornwinder in 1976. This is the first instance of using a vector space of polynomials/functions in the linear algebra method in combinatorics. Moreover, it can also be seen as one of the first instances of the so-called polynomial method in combinatorics.

Exercise 3 Let UD^n be the unit distance graph in $\mathbb{R}^n$. Prove that for n large enough, we have $\chi(\mathrm{UD}^n) \le n^{n/2} \cdot 2^n$.

Exercise 4 Let $SF(\ell, k)$ be the smallest integer N, such that every family of N sets of size ℓ contains a sunflower with k petals.

(*a*) Show that $SF(2,3) = 7$.

(*b*) Show that for every even ℓ, we have $SF(\ell, 3) > \sqrt{6}^{\ell}$

HINTS

Exercise 3: Partition the space into cubes of diameter 3/4.