

RECAP: Triangle-free subgraphs_____

Theorem. (Mantel, 1907) The maximum number of edges in an n -vertex **triangle-free** graph is $\lfloor \frac{n^2}{4} \rfloor$.

Proof.

(i) *There is a triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges.*

(ii) If G is a triangle-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Proof of (ii) is with extremality. (Look at the neighborhood of a vertex of maximum degree.)

Complete k -partite graphs

A graph G is **r -partite** (or **r -colorable**) if there is a partition $V_1 \cup \dots \cup V_r = V(G)$ of the vertex set, such that for every edge its endpoints are in *different* parts of the partition.

G is a **complete r -partite graph** if there is a partition $V_1 \cup \dots \cup V_r = V(G)$ of the vertex set, such that $uv \in E(G)$ iff u and v are in *different* parts of the partition. If $|V_i| = n_i$, then G is denoted by K_{n_1, \dots, n_r} .

The **Turán graph** $T_{n,r}$ is the complete r -partite graph on n vertices whose partite sets differ in size by at most 1. (All partite sets have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.)

Lemma Among **r -colorable** graphs the Turán graph is the *unique* graph, which has the most number of edges.

Proof. Local change. □

Turán's Theorem

The **Turán number** $ex(n, H)$ of a graph H is the largest integer m such that there exists an H -free* graph on n vertices with m edges.

Example: Mantel's Theorem states $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Theorem. (Turán, 1941)

$$ex(n, K_r) = e(T_{n,r-1}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + O(n).$$

Proof. Prove by induction on r that

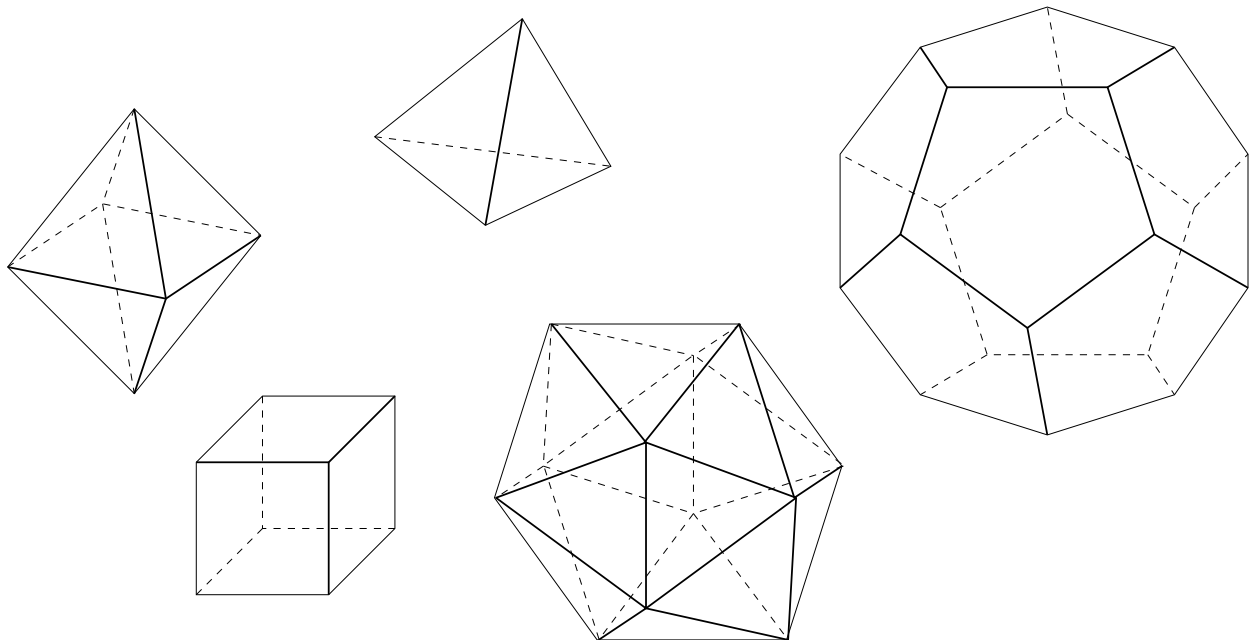
$G \not\supseteq K_r \implies$ **there is** an $(r-1)$ -partite graph H with $V(H) = V(G)$ and $e(H) \geq e(G)$.

Then apply the Lemma to finish the proof. □

*Here H -free means that there is no subgraph isomorphic to H

Turán-type problems

Question. (Turán, 1941) What happens if instead of K_4 , which is the graph of the tetrahedron, we forbid the graph of some other platonic polyhedra? How many edges can a graph without an octahedron (or cube, or dodecahedron or icosahedron) have?



The platonic solids

Erdős-Simonovits-Stone Theorem_____

Theorem. (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$

Corollary. (Erdős-Simonovits, 1966) For any graph H ,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2).$$

Corollaries of the Corollary.

$$ex(n, \text{octahedron}) = \frac{n^2}{4} + o(n^2)$$

$$ex(n, \text{dodecahedron}) = \frac{n^2}{4} + o(n^2)$$

$$ex(n, \text{icosahedron}) = \frac{n^2}{3} + o(n^2)$$

$$ex(n, \text{cube}) = o(n^2)$$

Proof of the Erdős-Simonovits Corollary_____

Theorem. (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$

Corollary. (Erdős-Simonovits, 1966) For any graph H ,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2).$$

Proof of the Corollary. Let $r = \chi(H)$.

- $\chi(T_{n,r-1}) < \chi(H)$, so $e(T_{n,r-1}) \leq ex(n, H)$.
- $T_{r\alpha,r} \supseteq H$, so $ex(n, T_{r\alpha,r}) \geq ex(n, H)$, where α is a constant depending on H ; say $\alpha = \alpha(H)$.

□

The number of edges in a C_4 -free graph_____

Theorem (Erdős, 1938) $ex(n, C_4) = O(n^{3/2})$

Proof. Let G be a C_4 -free graph on n vertices.

$C = C(G) :=$ number of $K_{1,2}$ (“cherries”) in G .

Doublecount C .

Counting by the midpoint: Every vertex v is the midpoint of exactly $\binom{d(v)}{2}$ cherries. Hence

$$C = \sum_{v \in V} \binom{d(v)}{2}.$$

Counting by the endpoints: Every pair $\{u, w\}$ of vertices form the endpoints of **at most one** cherry. (Otherwise there is a $C_4 \subseteq G$.) Hence

$$C \leq 1 \cdot \binom{n}{2}.$$

Proof cont'd

Combine and apply Jensen's inequality
(Note that $x \rightarrow \binom{x}{2}$ is a convex function)

$$\binom{n}{2} \geq C \geq \sum_{v \in V} \binom{d(v)}{2} \geq n \cdot \binom{\bar{d}(G)}{2}.$$

$\bar{d}(G) = \frac{1}{n} \sum_{v \in V} d(v)$ is the **average degree** of G .

$$\frac{n-1}{2} \geq \binom{\bar{d}(G)}{2} \geq \frac{(\bar{d}(G)-1)^2}{2}$$

Hence $\sqrt{n-1} + 1 \geq \bar{d}(G)$. □

Theorem (E. Klein, 1938) $ex(n, C_4) = \Theta(n^{3/2})$

Proof. Homework.

Theorem (Kővári-Sós-Turán, 1954) For $s \geq t \geq 1$

$$ex(n, K_{t,s}) \leq c_s n^{2-\frac{1}{t}}$$

Proof. Homework.

Open problems and Conjectures_____

Known results.

$$\Omega(n^{3/2}) \leq ex(n, Q_3) \leq O(n^{8/5})$$

$$\Omega(n^{9/8}) \leq ex(n, C_8) \leq O(n^{5/4})$$

$$\Omega(n^{5/3}) \leq ex(n, K_{4,4}) \leq O(n^{7/4})$$

Conjectures.

$$ex(n, K_{t,s}) = \Theta\left(n^{2 - \frac{1}{\min\{t,s\}}}\right) \text{ true for } t = 2, 3 \text{ and } s \geq t \\ \text{or } t \geq 4 \text{ and } s > (t - 1)!$$

$$ex(n, C_{2k}) = \Theta\left(n^{1 + \frac{1}{k}}\right) \text{ true for } k = 2, 3 \text{ and } 5$$

$$ex(n, Q_3) = \Theta\left(n^{\frac{8}{5}}\right)$$

If H is a d -degenerate bipartite graph, then

$$ex(n, H) = O\left(n^{2 - \frac{1}{d}}\right).$$