The Kruskal-Katona theorem

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1 Introduction

We start off this section by trying to characterize when equality happens in Sperner's Theorem. Which antichains have $\binom{n}{\lfloor n/2 \rfloor}$ members?

Recall our proof: If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then

$$1 \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

In order to have equality, we need to have $\binom{n}{|F|} = \binom{n}{\lfloor n/2 \rfloor}$ for all $F \in \mathcal{F}$. That is, all sets in $\mathcal{F}$ must have size either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. If n is even, this already characterizes the unique maximum antichain: $\binom{[n]}{n/2}$ For n odd, however there are two possible sizes for the members of $\mathcal{F}$ and there are at least two maximum antichains: $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$.

Are there more? Can we combine elements from the two middle levels of the boolean poset to make another maximum antichain? The answer to this question will be a simple consequence of the main result of this section: the Kruskal-Katona Theorem. One of the central theorems of extremal combinatorics with significant applications in algebraic combinatorics.

Let us just start slow. If $\mathcal{F}$ is a maximum antichain, then let $\mathcal{F}^* := \mathcal{F} \cap \binom{[n]}{\lceil n/2 \rceil}$ be the members of $\mathcal{F}$ of size $\lceil n/2 \rceil$. Then for sure we cannot anymore have a set of size $\lfloor n/2 \rfloor$ in $\mathcal{F}$ which is contained in a member of $\mathcal{F}^*$. Hence the following definition will be convenient.

Definition 1.1. *Given a k-uniform family $\mathcal{G} \subseteq \binom{[n]}{k}$, the* shadow *of $\mathcal{G}$, denoted by ∂G, is the $(k-1)$-uniform family*

$$\partial G := \left\{ F' \in \binom{[n]}{k-1} : F' \subset F \textit{ for some } F \in \mathcal{G} \right\}.$$

So antichain $\mathcal{F}$ cannot contain any sets from $\partial \mathcal{F}^*$. Since $\mathcal{F}$ is a maximal antichain and it only contains sets of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, it must be of the form

$$\mathcal{F} = \mathcal{F}^* \bigcup \left(\binom{[n]}{\lfloor n/2 \rfloor} \setminus \partial \mathcal{F}^* \right).$$

So $\mathcal{F}$ is of maximum size if $|\mathcal{F}^*| - |\partial \mathcal{F}^*|$ is as large as possible.

In fact, by Sperner's Theorem $|\mathcal{F}^*|$ can never be more than $|\partial \mathcal{F}^*|$, as that would imply $|\mathcal{F}| = |\mathcal{F}^*| + \binom{n}{\lfloor n/2 \rfloor} - |\partial \mathcal{F}^*| > \binom{n}{\lfloor n/2 \rfloor}$. So our question about the characterization can be reformulated about characterizing those $\lceil n/2 \rceil$-uniform families $\mathcal{F}^*$ whose shadow has size as small as $|\mathcal{F}^*|$. Of course this does happen for $\mathcal{F}^* = \binom{[n]}{\lceil n/2 \rceil}$ and for $\mathcal{F}^* = \emptyset$, but are there more examples?

We will study this question in even greater generality and will try to answer the following.

Question 1.2. *For given $n \geq k$, and m, $0 \leq m \leq \binom{n}{k}$, what is the smallest possible shadow a k-uniform family with m edges can have?*

1.1 A construction.

To get a feel for what could work, let us experiment with $k = 2$, i.e. with graphs. For a graph its shadow is the set of vertices that are *not* isolated. How can we minimize the non-isoltaed vertices? A cliqe seems like a good idea: once we touched a vertex with one edge, it would seem a waste to touch a new vertex before occupying all free edges from the new vertex to old vertices. Formally, we take the largest integer a_2 such that $m \geq \binom{a_2}{2}$, take a subset $A_2 \subseteq [n]$ of size a_2, and add a clique on A_2 in our construction. If $m = \binom{a_2}{2}$, we are done. Otherwise we still have $m - \binom{a_2}{2}$ edges to place, we can do that by adding one more vertex and connecting it to vertices of A_2. Note that this is always possible as $m - \binom{a_2}{2} < a_2$, as $m < \binom{a_2+1}{2} = \binom{a_2}{2} + a_2$. So we managed to find a construction with a shadow of size $a_2 + 1$. Also, one cannot do better since a_2 vertices can accommodate only $\binom{a_2}{2} < m$ edges, and for the remaining edges we need to add at least one more vertex.

Let us carry on with this idea for larger uniformity k. Given a target edge number m, $0 \leq m \leq \binom{n}{k}$, it seems like a smart idea to take first a clique of size as large as possible. Let a_k be the largest integer such that $m \geq \binom{a_k}{k}$, take a subset $A_k \subseteq [n]$ of size a_k, and add all k-subsets of A_k to our construction.

We still need to place $m - \binom{a_k}{k}$ edges. Since A_k hosts a clique, we are forced to involve a new vertex in our construction. and We plan to add k-sets that intersect A_k as much as possible, i.e. in $k-1$ vertices. Furthermore, in order to reduce the shadow of the k-sets containing the new vertex, these $(k-1)$-sets will be as densely packed as possible, i.e., they will form a clique within A_k.

Let a_{k-1} be the largest integer such that $\binom{a_{k-1}}{k-1} \leq m - \binom{a_k}{k}$. Note that $a_{k-1} < a_k$, since $m - \binom{a_k}{k} < \binom{a_k+1}{k} - \binom{a_k}{k} = \binom{a_k}{k-1}$. We choose a new vertex $v_k \in [n] \setminus A_k$, a subset $A_{k-1} \subset A_k$ of size a_{k-1}, and add all those k-sets to our construction which consists of v_k and a $(k-1)$-subset of A_{k-1}.

We are left with $m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1}$ edges to place. Now we are forced again to involve another vertex v_{k-1} in our construction. Note however, that we are not anymore forced to take an entirely new vertex for v_{k-1}, since $\binom{a_k+1}{k} > m$, so the k-sets of $A_k \cup \{v_k\}$ should be sufficient to select our remaining k-sets from. We hence choose a vertex $v_{k-1} \in [n] \setminus A_{k-1}$ (note that this is possible, since $a_{k-1} < a_k$). In order to minimize the shadow we make sure that the new k-sets we add also contain v_k, and the remaining $(k-2)$-sets they contain also form a clique within A_k. This way the $(k-1)$-sets containing v_k, but not v_{k-1} are already in the shadow anyhow. For this we choose an integer a_{k-2}, such that $\binom{a_{k-2}}{k-2} \leq m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1}$, and a subset $A_{k-2} \subset A_{k-1}$. Note that $a_{k-2} < a_{k-1}$, as $m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1} < \binom{a_{k-1}+1}{k-1} - \binom{a_{k-1}}{k-1} = \binom{a_{k-1}}{k-2}$.

In general we proceed similarly. We construct nested sets $A_{k+1} := [n] \supset A_k \supset A_{k-1} \supset \cdots \supset A_s$ and vertices $v_j \in A_{j+1} \setminus A_j$, for $j = k, \dots, s$, such that the size a_j of A_j is chosen to be the largest integer with $\binom{a_j}{j} \leq m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1} - \cdots - \binom{a_{j+1}}{j+1}$. Then $a_j < a_{j+1}$, so there is place in $A_{j+1} \setminus \{v_j\}$ to choose A_j. We stop whenever $\binom{a_s}{s} = m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1} - \cdots - \binom{a_{s+1}}{s+1}$ for some s, so there will be no more k-edge to place. Our construction is then

$$\mathcal{G} = \bigcup_{j=s}^{k} \{T \cup \{v_k, v_{k-1}, \dots, v_{j+1}\} : T \subseteq A_j, |T| = j\}.$$

That is in the jth step, for $j = k, k-1, \dots, s$ we add all k-sets containing $v_k, \dots, v_{j+1}$ and a j-subset of the set A_j. The number of sets in $\mathcal{G}$ is

$$\binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{s+1}}{s+1} + \binom{a_s}{s},$$

which, by our stopping rules is equal to m.

What is the size of the shadow $\partial\mathcal{G}$? There are $\binom{a_k}{k-1}$ members contained in A_k. All the remaining mebers contain v_k. The number of those, which contain v_k but not v_{k-1} is exactly $\binom{a_{k-1}}{k-2}$. And so on, the number of $(k-1)$-subsets in the shadow that contain $v_k, v_{k-1}, \ldots, v_{j+1}$, but not v_j is exactly $\binom{a_j}{j-1}$. Hence the size of the shadow is

$$|\partial\mathcal{G}| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_{s+1}}{s} + \binom{a_s}{s-1}.$$

Remark. 1. The expression of the integer m as the sum of binomial coefficients

$$\binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{s+1}}{s+1} + \binom{a_s}{s}$$

such that $a_k > a_{k-1} > \cdots > a_s \geq k$, is called the *$k$-cascade representation* of m. Within our construction we implicitely established the existence of a k-cascade representation for any m. This representation is also unique (HW)
2. Our construction above contains the k-sets in an initial segment of the so-called *colexicographic order* of finite subsets of $\mathbb{N}$. (See HW)

The Kruskal-Katona Theorem states that the previous construction is best possible, that is it gives the exact minimum size of the shadow of a k-uniform family for any size m.
Theorem 1.3 (Kruskal-Katona, (1963,1968))**.** *If $\mathcal{F} \subseteq \binom{[n]}{k}$, with*

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \binom{a_{k-2}}{k-2} + \cdots + \binom{a_s}{s},$$

then

$$|\partial\mathcal{F}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \binom{a_{k-2}}{k-3} + \cdots + \binom{a_s}{s-1}.$$

Our proof strategy will be to modify an arbitrary family $\mathcal{F}$ with m k-sets into a nicer, "compressed" family of the same size, such that the shadow does not increase. The shadow of this nicer family will be more transparent, which allows us to prove the lower bound for it. We present the proof in the next two sections.

2 Shifting

Arbitrary k-uniform families can be pretty wild and first we will want to make them "more orderly". Our main tool for this will be the *shifting operator*, which replaces occurrences of $i \neq 1$ in the sets of the family with 1—whenever this is possible without "losing sight" of any set from the family. The hope with performing such an operation is two-fold. On the one hand it seems to "compress" the family towards the element 1, and hence one would expect the shadow not to increase. On the other hand shifting seems to mould a family towards the colex construction of the previous section, where the more transparent structure made the bounding of the shadow accessible.
Definition 2.1. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $2 \leq i \leq n$. Define the* shift operator *$S_i^{\mathcal{F}}$ by letting*

$$S_i^{\mathcal{F}}(F) := \begin{cases} F \setminus \{i\} \cup \{1\} & \text{if } i \in F,\ 1 \notin F, \text{ and } F \setminus \{i\} \cup \{1\} \notin \mathcal{F}, \\ F & \text{otherwise} \end{cases}$$

for any $F \in \mathcal{F}$. Then we define $S_i(\mathcal{F}) := \left\{ S_i^{\mathcal{F}}(F) : F \in \mathcal{F} \right\}$.

If $S_i^{\mathcal{F}}(F) \neq F$, then we say that the set F *shifted.*

Let us see a concrete example how the shift operator functions.

Example 2.2. *Let* $\mathcal{F} = \{123, 134, 136, 236, 345\} \subset \binom{[6]}{3}$ *(we have suppressed interior set brackets). Then*

$$
\begin{aligned}
&S_2(\mathcal{F}) = \{123, 134, 136, 236, 345\}, \\
&S_3(\mathcal{F}) = \left\{123, 134, 136, \boxed{126}, \boxed{145}\right\}, \\
&S_4(\mathcal{F}) = \left\{123, 134, 136, 236, \boxed{135}\right\}, \\
&S_5(\mathcal{F}) = \{123, 134, 136, 236, 345\}, \text{ and} \\
&S_6(\mathcal{F}) = \{123, 134, 136, 236, 345\}.
\end{aligned}
$$

The boxed triples are the ones that are the results of shifts, the other triples remained the same by the shift.

Our first claim just verifies that our definition of the shift operation is a good one, in the sense that we do not "lose" any set from our family while shifting.

Claim 2.3. *For any* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $2 \le i \le n$*, the shift operator* $S_i^{(k)} : \mathcal{F} \to S_i(\mathcal{F})$ *is injective. In particular, we have*

$$|S_i(\mathcal{F})| = |\mathcal{F}|.$$

Proof. Suppose that for some $F_1, F_2 \in \mathcal{F}$ we have $S_i^{\mathcal{F}}(F_1) = S_i^{\mathcal{F}}(F_2)$.

First note that it is impossible that exactly one of them, say F_1 is shifted. Indeed, otherwise the shifted set $S_i^{\mathcal{F}}(F_1) = F_1 \setminus \{i\} \cup \{1\}$ was *not* in $\mathcal{F}$, while at the same time it is equal to $S_i^{\mathcal{F}}(F_2) = F_2$ which *is* in $\mathcal{F}$, a contradiction.

Otherwise we show that $F_1 = F_2$. If none of F_1 and F_2 are shifted then $F_1 = S_i^{\mathcal{F}}(F_1) = S_i^{\mathcal{F}}(F_2) = F_2$. If both of them shifted then we have $F_1 = S_i^{\mathcal{F}}(F_1) \setminus \{1\} \cup \{i\} = S_i^{\mathcal{F}}(F_2) \setminus \{1\} \cup \{i\} = F_2$. $\square$

We expect that exchanging the element 1 into sets of the family will make it "more compressed". In our second claim we make this feeling precise by deriving that the shadow after a shift is never larger than before the shift.

Claim 2.4. *For any* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $2 \le i \le n$ *we have*

$$\partial(S_i(\mathcal{F})) \subseteq S_i(\partial\mathcal{F})$$

In particular $|\partial(S_i(\mathcal{F}))| \le |\partial\mathcal{F}|$.

Proof. Let us first note that by Claim 1.6 the first part of our statement implies the second one:

$$|\partial(S_i(\mathcal{F}))| \le |S_i(\partial\mathcal{F})| = |\partial\mathcal{F}|.$$

Let $E \in \partial(S_i(\mathcal{F}))$ be arbitrary. We will show $E \in S_i(\partial\mathcal{F})$. First, note that $E = (S_i(F)) \setminus \{x\}$ for some $F \in \mathcal{F}$, and $x \in S_i(F)$. We proceed by cases.

Case 1. $1, i \notin S_i(F)$.

In this case, $S_i(F) = F$, since $1 \notin S_i(F)$. Thus, $E \subset F$, i.e., $E \in \partial\mathcal{F}$. Since $i \notin E$, we have that $S_i(E) = E$, and so $E \in S_i(\partial\mathcal{F})$.

Case 2. $1, i \in S_i(F)$.

In this case, we again have $S_i(F) = F$ (since $i \in S_i(F)$). As before, we then have $E \in \partial\mathcal{F}$. Now, if $x \ne 1$, we have $S_i(E) = E$, since $1 \in E$, and so $E \in S_i(\partial\mathcal{F})$, as desired. If $x = 1$, then $E' = E \setminus \{i\} \cup \{1\} \subset F$. Since $E' \in \partial\mathcal{F}$, we then have that E is blocked from shifting (for if E shifts, then $S_i(E) = E'$, which is absurd), and so $S_i(E) = E$. Thus, $E \in S_i(\partial\mathcal{F})$ as always.

Case 3. $1 \notin S_i(F)$ and $i \in S_i(F)$.

As in earlier cases, we have that $S_i(F) = F$, and so $E \in \partial\mathcal{F}$. However, since then $1 \notin F$ and $i \in F$, we must have that $F' = F \setminus \{i\} \cup \{1\} \in \mathcal{F}$. If $x = i$, then $S_i(E) = E$ as in previous cases, and we are done. If $x \neq i$, then we note that since $E' = F' \setminus \{x\} \in \partial\mathcal{F}$, and $E' = E \setminus \{i\} \cup 1$, that $S_i(E) = E$ in this subcase, and we are again happy.

Case 4. $1 \in S_i(F)$ and $i \notin S_i(F)$.

Now, since $i \notin E$, we have that $S_i(E) = E$, and so if $E \in \partial\mathcal{F}$, we are done. If $F = S_i(F)$, we get this immediately, so we may assume $F \neq S_i(F)$. So, we must have that $i \in F$, $1 \notin F$, and $F \setminus \{i\} \cup \{1\} \notin \mathcal{F}$. Now, if $x = 1$, we then have that $E \subset F$, and so $E \in \partial\mathcal{F}$, as desired. If $x \neq 1$, then we have that $E' = E \setminus \{1\} \cup \{i\} \subset F$, and so $E' \in \partial\mathcal{F}$. Since we may assume $E \notin \partial\mathcal{F}$ (or we're done), we then have that $E \in S_i(\partial\mathcal{F})$, since $E = S_i(E')$ in this subcase. □

2.1 Stable families

In our proof we plan to keep performing shifts on the given arbitrary k-uniform family $\mathcal{F}$ until no shift changes it anymore.

Definition 2.5. *A family $\mathcal{F}$ is called* stable *if $S_i(\mathcal{F}) = \mathcal{F}$ for all $i \geq 2$.*

Claim 2.6. *For any $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, there exists some stable $\mathcal{G} \subset \binom{\mathbb{N}}{k}$ such that $|\mathcal{G}| = |\mathcal{F}|$ and $|\partial\mathcal{G}| \leq |\partial\mathcal{F}|$.*

Proof. If $\mathcal{F}$ is stable, we are done, so assume not. Let $\mathcal{F}^{(1)} = S_i(\mathcal{F})$ for some i such that $\mathcal{F}^{(1)} \neq \mathcal{F}$. By the instability of $\mathcal{F}$, $\mathcal{F}^{(1)}$ must exist. By Claim 1.6, we have that $\left|\mathcal{F}^{(1)}\right| = |\mathcal{F}|$, and by Claim 1.7, we have that $\left|\partial\mathcal{F}^{(1)}\right| \leq |\partial\mathcal{F}|$. Thus, if $\mathcal{F}^{(1)}$ is stable, we have found the required $\mathcal{G}$. Otherwise, we may repeat process, say k times, to find a $\mathcal{F}^{(k)}$ such that $\left|\mathcal{F}^{(k)}\right| = |\mathcal{F}|$, and $\left|\partial\mathcal{F}^{(k)}\right| \leq |\partial\mathcal{F}|$. Moreover, the number of times we can perform this operation is finite (since we increase the number of sets containing 1 at each step), and so we must eventually find a stable family $\mathcal{F}^{(k)}$ for some finite k with the desired properties. □

The motivation for stable families and Claim 1.9 is that the first m sets of $\binom{\mathbb{N}}{k}$ in the colexicographic order (our construction) is a stable family, and it is easier to show that the shadow of this family has the necessary size. Not all stable families are of this form [1], so we unfortunately need a few more claims to prove the desired result.

Let $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, let $\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin F\}$, let $\mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\}$, and let $\mathcal{F}_1^- = \{F \setminus \{1\} : F \in \mathcal{F}_1\}$. Note that then $\mathcal{F}_1^- \subset \binom{\mathbb{N}}{k-1}$.

We will show that if $\mathcal{F}$ is stable, then a great deal of information about $\partial\mathcal{F}$ is encoded in the form and size of $\mathcal{F}_1^-$. In particular, the size of the shadow $\partial\mathcal{F}$ is completely controlled by $\mathcal{F}_1^-$. Eventually, we will use this to prove Theorem 1.3.

Claim 2.7. *If $\mathcal{F}$ is stable, then $\partial\mathcal{F}_0 \subseteq \mathcal{F}_1^-$.*

Proof. Let $E \in \partial\mathcal{F}_0$. Then $E = F \setminus \{x\}$ for some $F \in \mathcal{F}_0$ and $x \geq 2$. Since $\mathcal{F}$ is stable, we have $S_x^{\mathcal{F}}(F) = F$. Since $1 \notin F$, and $x \in F$, the only reason for F not shifting can be $F' = F \setminus \{x\} \cup \{1\}$ already being in $\mathcal{F}$. Since $1 \in F'$ we also have $F' \in \mathcal{F}_1$ and then $E = F' \setminus \{1\} \in \mathcal{F}_1^-$, as desired. □

Claim 2.8. *If $\mathcal{F}$ is stable, then $\partial\mathcal{F} = \mathcal{F}_1^- \cup \{E \cup \{1\} : E \in \partial\mathcal{F}_1^-\}$. In particular, $|\partial\mathcal{F}| = \left|\mathcal{F}_1^-\right| + \left|\partial\mathcal{F}_1^-\right|$.*

[1] for example, any collection of k-sets in which all sets contain 1 is stable

Proof. By the previous claim $\partial\mathcal{F}_0 \subseteq \mathcal{F}_1^- \subseteq \partial\mathcal{F}_1$, so $\partial\mathcal{F} = \partial\mathcal{F}_0 \cup \partial\mathcal{F}_1 = \partial\mathcal{F}_1$. Now note that $\mathcal{F}_1^-$ is defined to contain exactly those members of the shadow $\partial\mathcal{F}_1$ which do not contain 1. For those members $F \in \partial\mathcal{F}_1$ of the shadow that do contain 1 there must be some (in fact unique, by Claim 1.6) $x \neq 1$, such that $F \cup \{x\} \in \mathcal{F}_1$. But then $F = E \cup \{1\}$ for the set $E = (F \cup \{x\}) \setminus \{1\} \setminus \{x\} \in \partial\mathcal{F}_1^-$, as desired.

□

3 Putting the claims together

Proof of Theorem 1.3. The proof is by double induction, first on k and then on m.

The base case of $k = 1$ is simple. The 1-cascade expansion is $m = \binom{m}{1}$ and the shadow of a 1-uniform family is always $\{\emptyset\}$ having $\binom{m}{1-1} = 1$ member.

We assume $k \geq 2$ and prove our statement by induction on the cardinality m of $\mathcal{F}$. If $m = 1$, then k-cascade expansion is $m = \binom{k}{k}$ and the family consists of a single k-set F. Then the shadow indeed contains only the $\binom{k}{k-1}$ $(k-1)$-subsets of F.

From now on let us also assume that $m > 1$.

We may assume that $\mathcal{F}$ is stable, since otherwise we would take the stable family $\mathcal{G}$ with $|\mathcal{G}| = |\mathcal{F}|$, provided by Claim 1.9, show the appropriate lower bound on $|\partial\mathcal{G}|$ and use that $|\partial\mathcal{G}| \leq |\partial\mathcal{F}|$.

As above let $\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin F\}$, $\mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\}$, and $\mathcal{F}_1^- = \left\{F \in \binom{[n]\setminus 1}{k-1} : F \cup \{1\} \in \mathcal{F}_1\right\}$.

We claim that

$$\left|\mathcal{F}_1^-\right| \geq \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_s - 1}{s-1}$$

Indeed, otherwise

$$\begin{aligned}
|\mathcal{F}_0| &= |\mathcal{F}| - |\mathcal{F}_1| = |\mathcal{F}| - \left|\mathcal{F}_1^-\right| \\
&> \left(\binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}\right) - \left(\binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_s - 1}{s-1}\right) \\
&= \binom{a_k - 1}{k} + \binom{a_{k-1} - 1}{k-1} + \cdots + \binom{a_s - 1}{s}
\end{aligned}$$

But then, using Claim 1.10 and induction for the family $\mathcal{F}_0$ (note that $|\mathcal{F}_0| < |\mathcal{F}|$ since $\mathcal{F}$ is stable), we obtain

$$\left|\mathcal{F}_1^-\right| \geq |\partial\mathcal{F}_0| > \binom{a_k - 1}{k-1} + \cdots + \binom{a_s - 1}{s-1},$$

which is a contradiction. Therefore, applying Claim 1.11 and induction for the $(k-1)$-uniform family $\mathcal{F}_1^-$, we get

$$\begin{aligned}
|\partial\mathcal{F}| &= \left|\mathcal{F}_1^-\right| + \left|\partial\mathcal{F}_1^-\right| \\
&\geq \left(\binom{a_k - 1}{k-1} + \cdots + \binom{a_s - 1}{s-1}\right) + \left(\binom{a_k - 1}{k-2} + \cdots + \binom{a_s - 1}{s-2}\right) \\
&\geq \binom{a_k}{k-1} + \cdots + \binom{a_s}{s-1},
\end{aligned}$$

as desired. □