What can *not* make the chromatic number small: the girth

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The chromatic number of graphs is one of the oldest, most-studied graph parameters - and one of the computationally most difficult one. In the previous lectures we were concerned about how the (small) number of edges can make the chromatic number of k-graphs small. Here we will examine what does *not* make the chromatic number small. We will get a glimpse of how elusive the chromatic number is when one tries to approach it through *local* properties of the graph.
The first and easiest lower bound on $\chi(G)$ is the clique number $\omega(G)$ of G. Obviously if there is a clique of size k in a graph then any proper coloring of G must use at least k colors, one for each vertex of the clique.
The other direction, that is $\chi(G) \le \omega(G)$ is very far from being true in general. In other words, the lack of $(k+1)$-cliques does *not* make a graph k-colorable. A first example is C_5, which has clique number 2, but chromatic number 3. More generally, in the Discrete Math I course you have encountered Myczielski's Construction: a triangle free graph with chromatic number k, for arbitrary k. That is, even the lack of triangles does *not* make the chromatic number small. [1]
In a triangle-free graph the neighborhoods of each vertex is independent. This "local" information was not enough to bound the chromatic number. Here we carry this further. We will consider not only the neighborhoods of vertices, but neighborhoods of neighborhoods, and neighborhoods of neighborhoods of neighborhoods, and beyond. What if we require from a graph that for every vertex v, looking around in its iterated neighborhoods, all we can "see" from it within distance 100 is the tree rooted at v? There is no other, "extra" edge, an edge that would close a cycle (of length at most 201). Can we then say something about the chromatic number of such a graph? Can we bound it?
It turns out that we can't. There exist graphs without any cycle of length at most 201, but having chromatic number 10^{100}. Such a graph locally looks like a tree, but yet its "global" structure must be really complex, as one cannot color it properly with less than 10^{10} colors.
The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle.

Theorem 1 (Erdős, 1959). *For every $k, l \in \mathbb{N}$ there exists a graph G with $g(G) \ge \ell$ and $\chi(G) \ge k$.*

This theorem says that despite locally being as sparse as possible, a graph can still be convoluted enough to have a large chromatic number. We will use a random graph to prove the xistence of our construction with "positive probability". However the uniform random graph $G(n, 1/2)$ typically produces a very dense graph, so it has way too many short cycles. So the 'positive probability' will be very tiny and our tools can not detect it. We thus will consider $G(n,p)$, where each edge appears independently, with probability $p = p(n)$. With a tiny p, we likely get a much sparser graphs and it will be easier to find graphs like what we're looking for. Indeed, just as a sanity check, let us see how many edges can a graph with girth $g(G)$ have.

[1] Graphs G with the property $\chi(H) = \omega(H)$ holding for every *induced* subgraph $H \subseteq G$ are called *perfect*. (The requirement that the $\chi = \omega$ equality should be true not only for G but all induced subgraphs is there to avoid calling graphs with artificially blown-up clique number perfect.) Perfect graphs are rare, yet remarkably beautiful objects, with a rich history and connections to other fields of Discrete Mathematics. The Strong Perfect Graph Conjecture of Berge was a driving force of structural graph theory for several decades until its resolution by Chudnovsky, Robertson, Seymour and Thomas in 2006. It states the following characterization: a graph is perfect if and only if it does not contain an induced odd cycle of length at least 5 or the complement of an induced odd cycle of length at least 5. It is easy to see that none of these graphs are perfect, but the converse is a 178-page tour de force of structural graph theory in the *Annals of Mathematics*.

Theorem 2. *If a graph G on n vertices contains no cycle of length less than g for some $g \in \mathbb{N}$, then*

$$e(G) \le O\left(n^{1+\frac{1}{k}}\right),$$

where $k = \lfloor \frac{g-1}{2} \rfloor$.

Proof. **HW** $\square$

Proof of Theorem 1. We sample a graph $G \sim G(n,p)$.

Short cycles: Let X_i be the number of cycles of length i, and let $X = \sum_{i=3}^{\ell-1} X_i$ be the total number of "short" cycles, cycles that we would not like to have in our final construction. To compute their expected number, note that there are $\frac{n(n-1)\dots(n-i+1)}{2i} \le \frac{n^i}{2i}$ cycles of length i in K_n (where $2i$ represents the number of automorphisms of C_i). The probability that any such fixed cycle appears in G is p^i (since C_i has i edges in it) and hence

$$\mathbb{E}(X) = \sum_{i=3}^{\ell-1} \mathbb{E}(X_i) \le \sum_{i=3}^{\ell-1} (np)^i \le \frac{(np)^\ell - 1}{np-1} \le (np)^\ell,$$

where the last inequality is true if we have $np \ge 2$. If we would take $p = 2/n$, or in fact any constant times $1/n$, we will get a low expected value for X, but the expected number of edges in $G(n,p)$ will be $\Theta(n)$, which is far too few for us to be able prove a large chromatic number.
Instead, our plan will be to choose a slightly larger p hence allowing a few number of these short cycles and removing one vertex from each, but still keeping most of the vertices.
We choose $p = n^{\theta-1}$ with any $\theta, 0 < \theta < 1/\ell$, say $\theta = \frac{1}{\ell+1}$. With this choice of p, the expected number of short cycles is $\mathbb{E}(X) \le (np)^\ell = n^{\theta\ell} \ll n$. Then by Markov's Inequality the probability that G has more than $n/2$ short cycles is small:

$$\mathbb{P}(X \ge n/2) \le \frac{\mathbb{E}(X)}{n/2} \le 2n^{\theta\ell-1} \to 0. \tag{1}$$

Chromatic number via the independence number: We plan to bound the bound the chromatic number from below via the inequality $\chi(G) \ge n/\alpha(G)$ we covered in Discrete Math I. To this end we need to bound $\alpha(G)$ from above. Our argument will be a straightforward union bound applications (similar to our lower bound proof for the symmetric Ramsey number). For some $m = m(n) \to \infty$,

$$\mathbb{P}[\alpha(G) \ge m] = \mathbb{P}[\exists \text{ independent } I \subseteq V(G), |I| = m] \le \sum_{I \subseteq \binom{V(G)}{m}} \mathbb{P}[I \text{ is independent}] \tag{2}$$

$$= \binom{n}{m}(1-p)^{\binom{m}{2}} \le n^m e^{-p\frac{m(m-1)}{2}} = \left(ne^{-p\frac{m-1}{2}}\right)^m \to 0 \tag{3}$$

provided $p\frac{m-1}{2} > \ln n$. So let us take $m = \left\lceil \frac{3\ln n}{p} \right\rceil = \left\lceil \frac{3n\ln n}{n^\theta} \right\rceil$.
Alteration: For our choices of p, m we have shown that for a large enough n, both $\mathbb{P}(X \ge n/2) < 1/2$ and $\mathbb{P}(\alpha(G) \ge m) < 1/2$, where X is the total number of cycles of length at most $\ell - 1$ in G. Thus, with positive probability, **both** $X < 1/2$ and $\alpha(G) < m$ hold. Hence, there *exists* an n-vertex graph G^* with less than $n/2$ short cycles and $\alpha(G^*) < m$. Now delete one vertex from each short cycle to obtain a graph $\tilde{G} \subseteq G^*$ on at least $n/2$ vertices, with $g(\tilde{G}) \ge \ell$ and $\alpha(\tilde{G}) \le \alpha(G^*) < m$. Therefore,

$$\chi(\tilde{G}) \ge \frac{v(\tilde{G})}{\alpha(\tilde{G})} > \frac{n/2}{m} \ge \frac{n/2}{\lceil 3n\ln n/n^\theta \rceil} \ge \frac{n^\theta}{7\ln n} \ge k$$

for n sufficiently large.

$\square$

What we have learned in this section is that *local parameters*, that is small clique-number or large (but finite) girth will not force the chromatic number to be small. We can interpret this as an indication that the chromatic number is a *global* parameter of graphs, one cannot derive *any* information about it via knowledge of what happens on constant size vertex sets.
We also learned that (for hypergraphs at least) a *global parameter* like the number of edges *can* force the chromatic number to be small. Some of the most famous open problems of graph theory concern whether other, more global-looking generalizations of the clique number can cause the chromatic number to be small.
An early question in this direction was Hajós' Conjecture. One can define the *topological clique number* $\omega_{top}(G)$ as the largest k such that there exists k vertices in G together with $\binom{k}{2}$ pairwise internally disjoint paths connecting each pair of the k vertices. Such a subgraph is called a topological k-clique. Obviously a K_k cliques itself is also a topological k-clique. Hajós' Conjecture stated that $\omega_{top}(G) \geq \chi(G)$. In other words the lack of topological $(k+1)$-clique should imply that the graph is k-colorable. This conjecture was proved to be true for $k = 3$ by Dirac, but eventually turned out to be false in general, already for $k = 6$. As far as I know its status is open for $k = 4$ and 5.
A further weakening of Hajós' Conjecture is Hadwiger's Conjecture. For that let us define $\omega_{minor}(G)$ to be the largest k such that there is a K_k-minor in G (that is there exists k disjoint vertex sets, each inducing a connected subgraph of G, such that any two of them are connected by an edge). Obviously any topological k-clique is a K_k-minor. Hadwiger's Conjecture states that $\omega_{minor}(G) \geq \chi(G)$. In other words Hadwigers Conjecture states that the lack of K_{k+1}-minor should imply that the graph is k-colorable. This conjecture is known to be true up to $k \leq 5$, with the proofs of the last two cases being long and complicated. Hadwiger's Conjecture is open for all other values of k.