Extremal Combinatorics Wi 2017–18
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Some inequalities and estimates

We review some of the basic inequalities and estimates that will be useful in this course.

Jensen's inequalitiy: If $f : \mathbb{R} \mapsto \mathbb{R}$ is a convex function, then for any $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

AM-GM-HM [1]: For any $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n} \geq \left(\frac{x_1^{-1} + \cdots + x_n^{-1}}{n}\right)^{-1}.$$

We often use the following two inequalities for simplifying expressions involving probabilities.

$$1 - x \leq e^{-x}, \text{ for all } x \in \mathbb{R},$$

$$(1-p)^n \geq 1 - pn, \text{ for all } p \in [0,1] \text{ and } n \in \mathbb{N}.$$

For estimating factorials and binomial coefficients, Stirling's approximation is the most powerful thing we have:

$$n! = (1 + O(1/n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

But we usually use the following simpler estimates. Using $(1 + 1/k)^k \leq e$ for all $k \in \mathbb{N}$, we can show that

$$\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

For binomial coefficients we have

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

Sometimes, the simple bound $\binom{n}{k} \geq (n-k+1)^k/k!$ is also useful.

[1] this can be proved using Jensen's inequality applied to some well-chosen functions

The middle binomial coefficient can be estimated as follows.

$$\frac{2^{2k}}{2\sqrt{k}} \leq \binom{2k}{k} \leq \frac{2^{2k}}{\sqrt{2k}}.$$

Or, we can use Stirling's approximation to see the truth

$$\binom{2k}{k} = \frac{2^{2k}}{\sqrt{\pi k}}(1 + o(1))$$

.

Binomial coefficients and the entropy function

If $k = \Omega(n)$, then we can estimate the binomial coefficient $\binom{n}{k}$ as follows. Using Stirling's approximation, we can write

$$\binom{n}{k} = (1 + o(1))\sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k}$$

Taking log on both sides, we get

$$\log \binom{n}{k} = \log(1 + o(1)) + \log \sqrt{\frac{n}{2\pi k(n-k)}} + k \log \frac{n}{k} + (n-k) \log \frac{n}{n-k}.$$

Since both k and $n - k$ are linear in n, the terms involving them dominates, and we can write

$$\log \binom{n}{k} = (1 + o(1)) \left(\frac{k}{n} \log \frac{n}{k} + \frac{n-k}{n} \log \frac{n}{n-k} \right) n = (1 + o(1)) H \left(\frac{k}{n} \right) n.$$

where $H(p) = -p \log p - (1-p) \log(1-p)$ be the binary entropy function defined for all $p \in (0, 1)$. In particular, if $k = cn$, then we get

$$\binom{n}{k} = 2^{H(c)n(1+o(1))}.$$