Extremal Combinatorics Wi 2017–18
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Solution suggestions to the practice exam

Exercise 1 (1) For an integer $r \geq 2$ and integers $t_1, \dots, t_r \geq 2$, the number $R_r(t_1, \dots, t_r)$ is defined as the smallest positive integer n such that for every map $c : E(K_n) \mapsto [r]$, there exists an $i \in [r]$ and t_i vertices in K_n such that for all edges e in the complete graph K_{t_i} induced on these vertices, we have $c(e) = i$.
Alternatively, one can just write:

$$R_r(t_1, \dots, t_r) := \min \left\{ n : \forall c : E(K_n) \to [r] \exists i \in [r] \; \exists K \in \binom{V(K_n)}{t_i} \forall e \in \binom{K}{2} \; (c(e) = i) \right\}.$$

(2) Let $n = r^{1+\sum_{i=1}^r (t_i - 2)}$ and fix an arbitrary colouring $c : E(K_n) \mapsto [r]$. Let $k = 1 + \sum_{i=1}^r (t_i - 2)$. Let $S_0 = V(K_n)$ and arbitrarily pick $v_1 \in S_0$. Then there are $r^k - 1$ edges of the form $v_1 v$, since $|S_0| = r^k$, each coloured with one of the r colours. Therefore, by the pigeonhole principle there exists a colour $c_1 \in [r]$ and a set $S_1 \subseteq S_0 \setminus \{v_1\}$ such that $|S_1| \geq \lceil \frac{r^k - 1}{r} \rceil = r^{k-1}$ and $c(v_1 v) = c_1$ for all $v \in S_1$. Inductively, we construct for every $i = 1, \dots, k+1$ sets $S_0 \supset S_1 \supset \cdots \supset S_i$, vertices $v_1 \in S_0, \dots, v_i \in S_{i-1}$ and colours $c_1, \dots, c_i$ such that $|S_i| \geq r^{k-i}$ and $c(v_j v) = c_j$ for all $j \in \{1, \dots, i\}$ and $v \in S_j$. Given this, we perform the induction step by picking an arbitrary vertex $v_{i+1} \in S_i$, and a colour c_{i+1} for which the size of the inverse image $c^{-1}(c_{i+1})$ is of maximum size within the set of edges $\{v_{i+1} v : v \in S_i \setminus \{v_{i+1}\}\}$. Then the size of the set $S_{i+1} := \{v \in S_i \setminus \{v_{i+1}\} : c(v_i v) = c_{i+1}\}$ is at least $\frac{|S_i| - 1}{r} \geq \lceil \frac{r^{k-i} - 1}{r} \rceil$, and by definition $c(v_{i+1} v) = c_{i+1}$ for all $v \in S_{i+1}$.

By the k-th step we have vertices $v_1, \dots, v_k$, colours $c_1, \dots, c_k$, and a set S_k with $|S_k| \geq r^{k-k} = 1$. Let v_{k+1} be a vertex in S_k. Then by our process we have for all $1 \leq i < j \leq k+1$, $c(v_i, v_j) = c_i$. Define a colouring $c^* : \{v_1, \dots, v_k\} \mapsto [r]$ by $c^*(v_i) = c_i \in [r]$. Applying the General Pigeonhole Principle, where the pigeonholes are defined by the inverse images of the colouring c^*, and noting that $k = 1 + \sum_{i=1}^r (t_i - 2)$, we have a colour $c_i \in [r]$ and a subset $T \subseteq \{v_1, \dots, v_k\}$ of size $|T| = t_{c_i} - 1$ such that and $c^*(v) = c_i$ for all $v \in T$. Therefore, $T \cup \{v_{k+1}\}$ is a monochromatic clique of size t_{c_i} in the colour c_i.

Exercise 2 (1) Any 2-uniform hypergraph with at most 2 edges is a bipartite graph, and hence two-colourable. Therefore, $m_B(2) > 2$. The graph C_3 (the triangle) gives rise to a non-two-colourable 2-uniform hypergraph with 3 edges, proving that $m_B(2) \leq 3$. Therefore, $m_B(2) = 3$.

(2) We first give an example of a 3-uniform hypergraph on 7 edges which is non-two-colourable, proving that $m_B(3) \leq 7$. Consider the Fano plane (Draw!) and a proper two-colouring of it. Let R be the set of vertices in the larger color class (say the red class), so $|R| \geq 4$. Fix an arbitrary vertex $x \in R$ and look at the three edges through x. Each of them can contain at most 1 other vertex of R since there is no red edge. Therefore, $|R| \leq 1+3 = 4$, so in fact we must have $|R| = 4$. This shows that *every* edge through an arbitrary red vertex x must contain one more other red point. In particular no edge of the Fano plane contains exactly one red vertex. Each pair of the 4 red vertices determines a unique edge of the Fano plane, that is at most $\binom{4}{2} = 6$ edges could possible have a red vertex. Since the Fano plane has 7 edges, one of these must be compltely blue, which is a contradiction. Hence, the Fano plane is a 3-uniform hypergraph on 7 edges which cannot be two-coloured, proving that $m_B(3) \leq 7$.

We now prove that every 3-uniform hypergraph on 6 edges can be two-coloured, which will imply $m_B(3) > 6$. Let $\mathcal{H}$ be such a hypergraph. Pick a uniformly random permutation σ of the vertices of $\mathcal{H}$ and let $v_1, \ldots, v_n$ be the ordering of the vertices given by σ. Colour these vertices in this order, always coloring blue, unless it is the last vertex of an edge in which all the other vertices are already coloured blue, in which case colour the vertex red. This colouring by definition does not contain any blue monochromatic edges. Moreover, it can only colour an edge f red, if there exists an edge $e \neq f$ such that the last vertex of e is the first vertex of f. For arbitrary edges e, f of $\mathcal{H}$, let $E_{e,f}$ be the event that the last vertex of e is the first vertex of f in the ordering $v_1, \ldots, v_n$ defined by the random permutation σ. Then

$$\Pr\{\text{there is a monochromatic edge}\} \subseteq \bigcup_{e,f \in E(\mathcal{H})} \Pr(E_{e,f}).$$

Therefore, if $\Pr\left(\cup_{e,f} E_{e,f}\right) < 1$, then there must exist an ordering of the vertices in which the algorithm gives us a proper two-colouring. We will now prove this claim.

Note that

$$\Pr(E_{e,f}) = \begin{cases} 0 & \text{if } |e \cap f| \neq 1 \\ \frac{(3-1)!(3-1)!}{(2 \cdot 3-1)!} = \frac{1}{30} & \text{if } |e \cap f| = 1. \end{cases}$$

Let k denote the number of pairs of edges (e, f) for which we have $|e \cap f| = 1$. Then $\Pr(\cup E_{e,f}) \leq k/30$, by the union bound. Now for the sake of contradiction assume that this probability is equal to 1. Then we must have $k \geq 30$. That means $k = 30$ and all of the $6 \cdot 5 = 30$ (ordered) pairs of distinct edges intersect in exactly one vertex and $E_{e,f}$ are pairwise disjoint events. We show that this is impossible. Pick an edge e. Since e has 3 vertices, and there are 5 edges other than e, there must be a vertex $v \in e$ and edges f and f' such that $e \cap f = e \cap f' = \{v\}$. But then there is an ordering of the vertices in which v is the last vertex of e which is the first vertex of f and also the first vertex of f', which shows that $E_{e,f}$ is not disjoint from $E_{e,f'}$, a contradiction.

Exercise 3 (1) Let A, B be two disjoint sets of vertices of a graph G. Then the pair (A, B) is called *ϵ-regular* if

$$|d(A,B) - d(A',B')| \leq \epsilon$$

for every $A' \subseteq A, B' \subseteq B$ with $|A'| \geq \epsilon |A|$ and $|B'| \geq \epsilon |B|$, where

$$d(S,T) = \frac{|\{uv \in E(G) : u \in S, v \in T\}|}{|S||T|}.$$

(2) Suppose there exists a $Y \subseteq A$ with $|Y| \geq \epsilon |A|$ for which the set $\{x \in B : |N(x,Y)| \geq (d-\epsilon)|Y|\}$ has cardinality less than $(1-\epsilon)|B|$. Then the set $X = \{x \in B : |N(x,Y)| < (d-\epsilon)|Y|\}$ has cardinality at least $\epsilon|B|$. Since A, B is an ϵ-regular pair, we must have $|d(X,Y) - d(A,B)| \leq \epsilon$. On the other hand,

$$d(X,Y) = \frac{\sum_{x \in X} |N(x,Y)|}{|X||Y|} < \frac{\sum_{x \in X} (d-\epsilon)|Y|}{|X||Y|} = d - \epsilon,$$

by the definition of X. This gives $d(A,B) - d(X,Y) > \epsilon$, contradicting the ϵ-regularity.

Exercise 4 Given a positive integer n and a graph H, $\mathrm{ex}(n, H)$ is the largest number of edges that a graph G on n vertices can have without containing H as a subgraph.

Let G be an n-vertex graph that does not contain any C_4, and has $\mathrm{ex}(n, C_4)$ edges. Then the minimum degree of G is at least 1, since if there was an isolated vertex in G then we can add an edge between this vertex and any other vertex of G to obtain a larger C_4-free graph, contradicting the maximality of $\mathrm{ex}(n, C_4)$.

Let S be the number of copies of $K_{1,2}$ in G. For every vertex v, there are exactly $\binom{d(v)}{2}$ copies of $K_{1,2}$ with v as their middle vertex.Therefore $S = \sum_{v \in V(G)} \binom{d(v)}{2}$.

On the other hand for every pair of vertices u, v in G there is at most one copy of $K_{1,2}$ having u and v as its leaves, because two such copies would create a copy of C_4 in G. Hence S is upper bounded by the number $\binom{n}{2}$ of pairs of vertices of G.

This gives us

$$\sum_{v \in V} \binom{d(v)}{2} \leq \binom{n}{2}.$$

Applying Jensen's inequality to the convex function $f(x) = x(x-1)/2$ we get

$$n\binom{\bar{d}}{2} \leq \sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n}{2},$$

where $\bar{d} = (\sum_{v \in V(G)} d(v))/n$ denotes the average degree of G. Then we have

$$\frac{n-1}{2} \geq \binom{\bar{d}}{2} = \frac{\bar{d}(\bar{d}-1)}{2} \geq \frac{1}{2}(\bar{d}-1)^2,$$

where in the last inequality we used that $\bar{d} \geq 1$ (since the minimum degree is at least 1). Expressing $\bar{d}$ we get $\bar{d} \leq \sqrt{n-1}+1$, which gives

$$e(G) \leq \frac{n\bar{d}}{2} = O(n^{3/2}),$$

by the Handshake Lemma.

Exercise 5 Denote the people of the town by elements of $[n] = \{1, \ldots, n\}$ and clubs that they form by subsets $S_1, \ldots, S_m$ of $[n]$.

We create an auxiliary family $S'_1 \ldots S'_m \subseteq [n+1]$, where $S'_i = S_i \cup \{n+1\}$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}_2^{n+1}$ denote the characteristic vectors of S'_i.

(1) We show that this set family satisfies the conditions of the Oddtown Theorem (from the lecture) and hence their number m is most the number $n+1$ of points of the base set. Indeed,

- since $|S_i|$ is even for all i, $|S'_i| = |S_i| + 1$ is odd for all i and
- since $|S_i \cap S_j|$ is odd for all $i \neq j$, $|S'_i \cap S'_j| = |S_i \cap S_j| + 1$ is even for all $i \neq j$.

As a constriction, we can take the $n-1$ sets $S_i := \{i, n\} \subseteq [n]$ for $1 \leq i \leq n-1$. Indeed, $|S_i| = 2$ is even for all i, and $|S_i \cap S_j| = 1$ is odd for all $i \neq j$.

(2) Let now n be odd. We claim that the family $\{S'_i : i = 1, \ldots, m\}$ together with the set $S'_{m+1} = [n]$ still satisfies the Oddtown rules and hence $m+1 \leq n+1$, implying $m \leq n$. For that it is enough to check the new restrictions: $|S'_{m+1}| = n$ is odd and $|S'_{m+1} \cap S'_i| = |S_i|$ is even.

Taking $S_i := \{i, n\}$ for $1 \leq i \leq n-1$ and $S_n = \{1, \ldots, n-1\}$ gives us a construction of n clubs, such that $|S_i|$ is even for every $i \in [n]$ and $|S_i \cap S_j| = 1$ is odd for every $i, j \in [n], i \neq j$.

Exercise 6 Let $V = \{v_1, \ldots, v_8\}$ denote the set of batteries. Then each trial can be denoted by a 3-element subset of v_i's. Let $V_0 = \{v_1, v_2, v_3\}$, $V_1 = \{v_4, v_5, v_6\}$ and $V_2 = \{v_7, v_8\}$. Define $E := \{e \in \binom{V}{3} : |e \cap V_i| = 2 \text{ and } |e \cap V_{i+1}| = 1 \text{ for } i \in \mathbb{Z}/3\mathbb{Z}\} \cup \{V_0, V_1\}$. Then $|E| = \binom{3}{2} \cdot 3 + \binom{3}{2} \cdot 2 + \binom{2}{2} \cdot 3 + 2 = 20$. We claim that if E is the set of trials that Tajel performs, then she can always ensure that she finds a working triple. So say Tajel has tested all of these 20 triples. Let w_1, w_2, w_3, w_4 be the working batteries among the 8 ones. If any three of these were placed in V_0 or V_1, then Tajel would have checked them and hence gotten a working triple. Therefore, each V_i contains at most 2 of the working batteries. Moreover, since there are four working batteries, there must exist an $i \in \mathbb{Z}/3\mathbb{Z}$ for which V_i contains *exactly* two working batteries, say w_1, w_2. Then if $\{w_3, w_4\} \cap V_{i+1} \neq \emptyset$, Tajel would have checked the triple $\{w_1, w_2, x\}$ in her trials where $x \in \{w_3, w_4\} \cap V_{i+1}$. Otherwise $\{w_3, w_4\} \cap V_{i+1} = \emptyset$ and consequently we must have $\{w_3, w_4\} \subseteq V_{i-1}$. In this case the triple $\{w_3, w_4, w_1\}$ is in E, and hence Tajel would have checked it. Thus we deduce that irrespective of how w_i's are distributed, Tajel would have found a working triple in these 20 trials.

Remark The construction is in fact the complement of the construction we gave in the lectures for the $K_4^{(3)}$ Turán problem. The strategy of Tajel is to build a 3-uniform hypergraph on 8 vertices whose complement does *not* contain any copy of $K_4^{(3)}$ (the hypergraph formed by taking all 3-subsets of a 4-set), which will ensure that for any four vertices there is a triple that forms an edge of her hypergraph. Therefore any lower bound on $\mathrm{ex}(n, K_4^{(3)})$ will give an upper bound on the best that Tajel can do.