The Erdős-Turán conjecture

A set S of positive integers is k-AP-free if $\{a, a+d, a+2d, \ldots, a+(k-1)d\} \subseteq S$ implies $d = 0$.

$s_k(n) = \max\{|S| : S \subseteq [n] \text{ is } k\text{-AP-free}\}$

How large is $s_k(n)$? Could it be linear in n?

Erdős-Turán Conjecture (Szemerédi's Theorem)
For every constant k, we have

$$s_k(n) = o(n).$$

Construction (Erdős-Turán, 1936)

$$s_3(n) \geq n^{\frac{\log 2}{\log 3}}.$$

$S = \{s : \text{ there is no 2 in the ternary expansion of } s\}$

S is 3-AP-free. For $n = 3^l$, $|S \cap [n]| = 2^l$

Roth's Theorem (1953) $s_3(n) = o(n)$.

History of Szemerédi's Theorem

Szemerédi's Theorem (1975) For any integer $k \geq 1$ and $\delta > 0$ there is an integer $N = N(k, \delta)$ such that any subset $S \subseteq \{1, \ldots, N\}$ with $|S| \geq \delta N$ contains an arithmetic progression of length k.

Was conjectured by Erdős and Turán (1936).
Featured problem in mathematics, inspired lots of great new ideas and research in various fields;

- Case of $k = 3$: analytic number theory (Roth, 1953; Fields medal)
- First proof for arbitrary k: combinatorial (Szemerédi, 1975)
- Second proof: ergodic theory (Furstenberg, 1977)
- Third proof: analytic number theory (Gowers, 2001; Fields medal)
- Fourth proof: fully combinatorial (Rödl-Schacht, Gowers, 2007)
- Fifth proof: measure theory (Elek-Szegedy, 2007+)

One of the ingredients in the proof of Green and Tao: "primes contain arbitrary long arithmetic progression"

Applications of the Regularity Lemma

Removal Lemma For $\forall\gamma > 0$ $\exists\delta = \delta(\gamma)$ such that the following holds. Let G be an n-vertex graph such that at least $\gamma\binom{n}{2}$ edges has to be deleted from G to make it triangle-free. Then G has at least $\delta\binom{n}{3}$ triangles.

Proof. Apply Regularity Lemma (Homework).

Roth's Theorem For $\forall\epsilon > 0$ $\exists N = N(\epsilon)$ such that for any $n \geq N$ and $S \subseteq [n]$, $|S| \geq \epsilon n$,
there is a three-element arithmetic progression in S.

Proof. Create a tri-partite graph $H = H(S)$ from S.

$$V(H) = \{(i,1) : i \in [n]\} \cup \{(j,2) : j \in [2n]\} \cup \{(k,3) : k \in [3n]\}$$

$(i,1)$ and $(j,2)$ are adjacent if $j - i \in S$
$(j,2)$ and $(k,3)$ are adjacent if $k - j \in S$
$(i,1)$ and $(k,3)$ are adjacent if $k - i \in 2S$

Roth's Theorem — Proof cont'd

$(i, 1), (i + x, 2), (i + 2x, 3)$ form a triangle for every $i \in [n]$, $x \in S$.
These $|S|n$ triangles are pairwise edge-disjoint.

$$\Downarrow$$

At least $\epsilon n^2 \geq \frac{\epsilon}{18}\binom{|V(H)|}{2}$ edges must be removed from H to make it triangle-free.

Let $\delta = \delta\left(\frac{\epsilon}{18}\right)$ provided by the Removal Lemma.
There are at least $\delta\binom{|V(H)|}{3}$ triangles in H.

S has no three term arithmetic progression

$$\Downarrow$$

$\{(i, 1), (j, 2), (k, 3)\}$ is a triangle iff $j - i = k - j \in S$.
Hence the number of triangles in H is equal to $n|S| \leq n^2 < \delta\binom{6n}{3}$, provided $n > N(\epsilon) := \left\lfloor \frac{1}{\delta} \right\rfloor$. $\square$

Behrend's Construction

Construction (Behrend, 1946)

$$s_3(n) \geq n^{1-O\left(\frac{1}{\sqrt{\log N}}\right)}.$$

Construct set of vectors $\bar{a} = (a_0, a_1, \ldots, a_{l-1})$:

$V_k = \{\bar{a} \in \mathbb{Z}^l \;\;:\;\; \|\bar{a}\|^2 = k,\; 0 \leq a_i < \frac{d}{2}$ for all $i < q\}$,

where $\|\bar{a}\| = \sqrt{\sum_{i=0}^{l-1} a_i^2}$.

Interpret a vector $\bar{a} \in \{0, 1, \ldots, d-1\}^l$ as an integer written in d-ary:

$$n_{\bar{a}} = \sum_{i=0}^{l-1} a_i d^i.$$

Let

$$S_k = \{n_{\bar{a}} : \bar{a} \in V_k\}$$

Claim $S_k \subseteq [d^l]$ is 3-AP-free for every k.

Proof. Assume $n_{\bar{a}} + n_{\bar{b}} = 2n_{\bar{c}}$.
Then $a_i + b_i = 2c_i$ for every $i < l$, because $a_i + b_i$ and $2c_i$ are both $< d$ (so there is no carry-over)
So $\bar{a} + \bar{b} = 2\bar{c}$. But

$$\|2\bar{c}\| = 2\|\bar{c}\| = 2\sqrt{k} = \|\bar{a}\| + \|\bar{b}\| \geq \|\bar{a} + \bar{b}\|,$$

and equality happens only if $\bar{a}$ and $\bar{b}$ are parallel. Since they are of the same length, we conclude $\bar{a} = \bar{b}$. □

Take the *largest* S_k. Bound its size by averaging:

$\bar{a} \in \{0, 1, \ldots, d-1\}^l \Rightarrow \|\bar{a}\|^2 < ld^2$,
so there is a k for which

$$|S_k| \geq \frac{|\bigcup_i S_i|}{ld^2} = \frac{(d/2)^l}{ld^2} = \frac{d^{l-2}}{2^l l}$$

For given N, choose $l = \sqrt{\log N}$ and $d = N^{\frac{1}{l}}$.