Hypergraph Ramsey theory

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1 A motivation: the Happy Ending Problem

> Our present problem has been suggested by Miss Esther Klein in connection with the following proposition.
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> From 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining a convex quadrilateral.
>
> We present E. Klein's proof here because later on we are going to make use of it. If the least convex polygon which encloses the points is a quadrilateral or a pentagon the theorem is trivial. Let therefore the enclosing polygon be a triangle ABC. Then the two remaining points D and E are inside ABC. Two of the given points (say A and C) must lie on the same side of the connecting straight line $\overline{DE}$. Then it is clear that $AEDC$ is a convex quadrilateral.
>
> Miss Klein suggested the following more general problem. *Can we find for a given n a number N such that from any set containing at least N points it is possible to select n points forming a convex polygon?*
>
> There are two particular questions: (1) does the number N corresponding to n exist? (2) If so, how is the least $N(n)$ detremined as a function of n?

The text above is from the introduction of a paper of Paul Erdős and George Szekeres from 1935. We put it here in original quote, because in retrospect this paper turned out be pioneering in two different, and at the time completely new fields: *combinatorial geometry* and *Ramsey theory*.
Two years before, in 1933, the three main protagonists, along with a group of other young mathematically inclined, like Turán (whose name we will also hear a lot this semester), were meeting regularly after university in the main park of Budapest and taking long walks in the wood to discuss, what else, mathematics. Apparently, already then, this is what the cool kids were doing in their free time.
It was at one of these meetings when Esther confronted the boys with her proof for a convex quadrangle and the general question. Paul and George immediately jumped on the topic with great enthusiasm, they got really excited by what they felt was a completely new type of geometric problem. They gave constructions of point sets of size 2^{t-2} in general position, without containing a convex t-gon. They also answered (1) in two different ways and in the process they rediscovered Ramsey's Theorem independently.
To motivate how the connection might have come about let us first make a few things precise from the introduction. First of all we will be dealing with point sets P that are *in general position*, i.e., no three points of P are on the same line. With forsight we will denote by $HE(t)$ (and not $N(t)$) the minimum integer n such that any n points in the plane in general position contain t points spanning a convex t-gon.
All four-element point sets in general position in the plane are exactly one of two kinds: either they form a convex 4-gon or not, depending on whether their convex hull is a quadrilateral or a triangle. The first easy, but important, observation is that the convexity of a t-element set in general position could be characterized through the convexity of its four-element subsets.

Proposition 1.1. *A t-element subset in general position forms a convex t-gon if and only if all $\binom{t}{4}$ of its four-subsets form a convex 4-gon.*

Proof. If $P \subseteq \mathbb{R}^2$ forms a convex t-gon, then no point is the convex combination of the other $n-1$. In particular no point is the convex combination any other three points, so every four-subset is convex.
In the other direction, suppose that every four-subset of P is a convex 4-gon. If a point $p \in P$ would be a convex combination of the others, then it is also a convex combination of just three of them: the vertices of the triangle which contains it, from an arbitrary triangulation of the convex hull of P. This provides non-convex four-subset of P, a contradiction. $\square$

The second important observation is the proposition of Klein, which says that it is impossible that for some five-element set *none of* the $\binom{5}{4}$ four-element subsets are convex.

Proposition 1.2. *It cannot happen that for some* 5*-element point set in general position none of the four-element subset forms a convex* 4*-gon.*

The natural classification of four-element point sets and the relation of these classes to larger point sets lead Erdős and Szekeres to the idea to *color the four-element subsets* of n points in general position by `red` or `blue` given whether they are in convex position or not, respectively. Then Proposition 1.1 translates to a t-element subset being in convex position if and only if all its $\binom{t}{4}$ four-element subset are `red`. Klein's proposition on the other hand forbids the presence of a five-element set with all its four-subsets being `blue`.
So, let's do some wishful thinking. If we were to know that there exists an integer, however large, but finite, denoted mysteriously by $R^{(4)}(t,5)$, such that for *any* `red`/`blue`-coloring $c : \binom{[R]}{4} \to \{\texttt{red}, \texttt{blue}\}$ of the 4-element subsets of the $R^{(4)}(t,5) =: R$-element set $[R]$, there exists a t-element subset $T \subseteq [R]$ with all its 4-subsets `red` *or* a 5-element subset $T \subseteq [R]$ with all its 4-subsets `blue`, so if we know all this, then we would be done! Because then, we claim, $HE(t) \leq R^{(4)}(t,5)$, so $HE(t)$ was also finite. Indeed, should such miraculous $R := R^{(4)}(t,5)$ existed for some t, then taking an arbitrary set $P \subseteq \mathbb{R}^2$ of R points in general position and creating the coloring described above, this coloring cannot contain a 5-subset with all its four-subsets having color `blue`! But then, by the magic property of the number R, there must be a t-subset $T \subseteq [R]$ for which *every* 4-subset is `red`. And that, via Proposition 1.1, implies that T is in convex position!
All we need is the existence of such a magic number $R^{(4)}(t,5)$. This motivated Erdős and Szekeres,[1] and motivates us as well, to introduce a Ramsey number for colorings, where instead of edges (i.e. 2-element subsets), we color k-element subsets.

2 The hypergraph Ramsey theorem

What is a hypergraph? It is a generalization of the concept of graphs, where instead of just 2-element vertex sets, as edges, we consider arbitrary subsets of a vertex set V. Formally, a *hypergraph* is defined as a pair $(V, \mathcal{F})$ of a vertex set V and edge set $\mathcal{F}$, where $\mathcal{F} \subseteq 2^V$. Often, if it is not ambiguous, we omit referring to the vertex set and identify the hypergraph with its edge set $\mathcal{F}$. A hypergraph is called k*-uniform*, for some positive integer k, if all its edges have size k, that is if $\mathcal{F} \subseteq \binom{V}{k}$. A k-uniform hypergraph is sometimes called a k*-graph*. The edges of a hypergraph are sometimes called hyperedges, and the edges of a k-graph are sometimes called k-edges.

Examples.

(1) For $k = 2$, we get back our good old graph concept: a 2-graph is just a graph.

(2) The analogue of complete graphs: the *complete* k*-graph* on t vertices contains all k-subsets of the t-element vertex set $[t]$ and is denoted by $K_t^{(k)}$. In other words, $K_t^{(k)} = \left([t], \binom{[t]}{k}\right)$.

[1] There might also have been other motivating factors ... But this is just speculation .... Anyway, Esther Klein and George Szekeres were married a couple of years after the initiation of the problem by the former and its extension (together with Erdős) by the later. This prompted Paul Erdős to coin the term *Happy End Problem.* Klein and Szekeres escaped persecution of Jews in Hungary before the second world war and settled in Australia afterwards. They died within an hour of each other at the age of 95 and 94, respectively. A good example of how far an innocent-looking math problem might lead you ...

(3) There are various analogues of many graph theoretic concepts, like path and cycles: tight paths/cycles, loose paths/cycles, ℓ-tight paths/cycles, Berge-cycles, etc ...

(4) Projective planes: $V =$ points := 1-dimensional subspaces of K^3, $\mathcal{F} =$ lines := 2-dimensional subspace of K^3, where K is an arbitrary field. When K is the finite field $\mathbb{F}_q$, we get a $(q+1)$-uniform hypergraph with q^2+q+1 vertices and equally many hyperedges. E.g. the Fano plane is the 3-uniform hypergraph on 7 vertices with 7 edges corresponding to the projective plane over $\mathbb{F}_2$.

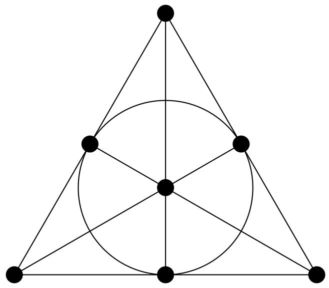

Figure 1: The Fano plane

For the rest of this section we will only be concerned with the complete k-uniform hypergraph. We will define hypergraph Ramsey number as the straightforward generalization of graph Ramsey numbers.

Definition 2.1 (Hypergraph Ramsey Number)**.** *Given $k \in \mathbb{N}$, and $s,t \geq k$, $R^{(k)}(s,t)$ is the minimum $n \in \mathbb{N}$ such that for every coloring $c : \binom{[n]}{k} \to$ `red`/`blue` there exists a set $T \in \binom{[n]}{t}$ such that $c(S) =$ `red` for every $S \in \binom{T}{k}$ or there exists a set $T \in \binom{[n]}{s}$ such that $c(S) =$ `blue` for every $S \in \binom{T}{k}$.*

Sometimes we refer to the property of the subset T in the definition that T hosts a `red` $K_s^{(k)}$ or that it hosts a `blue` $K_t^{(k)}$. We call a hypergraph with all its edges colored with the same color *monochromatic.*
Before going on on, let us make some simple observations:

(1) $R^{(2)}(s,t) = R(s,t)$.

(2) $R^{(1)}(s,t) = s+t-1$ (think over the detailed proof!)

(3) $R^{(k)}(s,t) = R^{(k)}(t,s)$.

(4) $R^{(k)}(k,t) = t$.

Analogous to the graph case, the first question we should ask ourselves is whether $R^{(k)}(s,t)$ is finite.

Theorem 2.2. *For arbitrary positive integers k, and $t,s \geq k$, the value $R^{(k)}(s,t)$ is finite.*

Remark. In the homework exercise you will be asked to define hypergraph Ramsey numbers for more than two colors and prove their finiteness. Moreover, the theorem also extends to the infinite setting analogously to the infinite Ramsey theorem for graphs.

First proof. We will follow the idea of the proof of Theorem 2.2 from week 1. and build a sequence where the color of every edge depends only on its smallest vertex in the sequence, and this way one can naturally identify a color with each vertex of the sequence. Then the 1-uniform Ramsey theorem will be used to select a subsequence where all these colors are the same, hence providing us with the monochromatic clique we want.

A sequence $v_1, \dots, v_\ell \in U$ of vertices will be called right-neighborhood-monochromatic in the set U, if the color of a k-set contained in U depends only on its element from the sequence with the smallest index (provided such an element exists). Formally, a sequence $v_1, \dots, v_\ell \in U$ is called *right-neighborhood-monochromatic in* U if there exists a coloring $c^* : [\ell] \to \{\texttt{red}, \texttt{blue}\}$ such that for every k-set $T \in \binom{U}{k}$ with $T \cap \{v_1, \dots, v_\ell\} \neq \emptyset$, we have $c(T) = c^*(v_{\min T})$, where we adopted the notation $\min T = \min\{j : v_j \in T\}$.

Similarly to Theorem 2.2 from week 1, our goal is to build a long enough right-neighborhood-monochromatic sequence $v_1, \dots, v_\ell$ in some set V_ℓ of large enough size. For us the length $\ell = t + s - 2k + 1$ will suffice with $|V_{t+s-2k+1}| = t + s - k$. Indeed, if we succeed to build such a sequence, then the 1-uniform Ramsey theorem will provide us with a subsequence of length $t-k+1$ which is c^*-monochromatic in $\texttt{red}$ or a subsequence of length $s-k+1$ which is c^*-monochromatic in $\texttt{blue}$. Such a sequence unioned with the remaining $k-1$ vertices in $V_{t+s-2k+1} \setminus \{v_1, \dots, v_{t+s-2k+1}\}$ forms a c-monochromatic subset T of the size required for its color. It is easy to check that a set T obtained this way is always monochromatic. For this just note that every k-subset $S \subseteq T$ does contain at least one element of the sequence (since the rest has only $k-1$ elements). Then the c-color of this S is the c^*-value of the minimum sequence-index appearing in S. But all these colors are the same by the way we selcted the elements of T from the sequence.

Let us see now how can we build recursively a right-neighborhood sequence $v_1, \dots, v_i$ in some set V_i and the appropriate coloring $c^* : [i] \to \{\texttt{red}, \texttt{blue}\}$.

For v_1 we choose an arbitrary vertex in $V_0 := [n]$. To find V_1, we consider the coloring of the complete $(k-1)$-uniform hypergraph on $V_0 \setminus \{v_1\}$ induced by the c-colors of the k-sets containing v. Formally, let $\tilde{c}(Q) := c(Q \cup \{v_1\})$ for every $Q \in \binom{V_0 \setminus \{v_1\}}{k-1}$. The $(k-1)$-uniform Ramsey theorem (true by induction) will provide us with a (large) $\tilde{c}$-monochromatic subset N_1 of size n_1 and we put $V_1 = N_1 \cup \{v_1\}$. Then v_1 is right-neighborhood-monochromatic in V_1 by the definition of N_1, because we can simply define $c^*(1)$ to be the $\tilde{c}$-color of the $(k-1)$-subsets of N_1.

Given a sequence $v_1, \dots, v_i$ that is right-neighborhood-monochromatic in some V_i with an appropriate function $c^* : [i] \to \{\texttt{red}, \texttt{blue}\}$ we choose v_{i+1} arbitrarily from $N_i := V_i \setminus \{v_1, \dots, v_i\}$. We find V_{i+1} by considering the coloring $\tilde{c}$, defined by $\tilde{c}(Q) := c(Q \cup \{v_{i+1}\})$ for every $Q \in \binom{N_i \setminus \{v_{i+1}\}}{k-1}$, and (hoping to) use the $(k-1)$-uniform Ramsey theorem to provide us with a (large) $\tilde{c}$-monochromatic subset N_{i+1} of size n_{i+1}. Then we put $V_{i+1} = N_{i+1} \cup \{v_1, \dots, v_{i+1}\}$. The vertex v_{i+1} is right-neighborhood-monochromatic in $N_{i+1} \cup \{v_{i+1}\}$ because N_{i+1} is $\tilde{c}$-monochromatic and consequently the whole sequence $v_1, \dots v_{i+1}$ is right-neighborhood-monochromatic in V_{i+1}. Indeed, we can simply extend the already existing $c^* : [i] \to \{\texttt{red}, \texttt{blue}\}$ to $i+1$ by defining $c^*(i+1)$ to be the $\tilde{c}$-color of the $(k-1)$-subsets of N_{i+1}.

Now the only thing left to do is to make sure that we are able to build a long enough sequence and have the last set $V_{t+s-2k+1}$ of size at least $k-1$. We secure this by choosing the sizes n_i large enough with respect to n_{i+1}. We set $n_{t+s-2k+1} = k-1$. To make sure that there is a monochromatic set N_{i+1} of size n_{i+1} in an arbitrary $\texttt{red}/\texttt{blue}$-coloring of the $(k-1)$-subsets of a set of size n_i we recursively choose $n_i = 1 + R^{(k-1)}(n_{i+1}, n_{i+1})$ for every $i = s+t-2k, \dots, 2, 1, 0$. We can do this because by induction all these $(k-1)$-uniform Ramsey numbers are finite. Consequently we succed in creating the appropriate right-neighborhood-monochromatic sequence, provided $n \geq n_0 + 1$. $\square$

Homework. *Prove the recursion* $R^{(k)}(t,s) \leq R^{(k-1)}(R^{(k)}(t-1,s), R^{(k)}(t,s-1))$ *and conclude Theorem 2.2.*

An immediate question after verifying any kind of finiteness should be “How large is finite?” Let us see. Using the recursion $n_i = R^{(k-1)}(n_{i+1}, n_{i+1}) + 1 =: R^{(k-1)}(n_{i+1}) + 1$ and the final value $n_{s+t-2k+1} = k-1$, we obtain

$$n_0 = R^{(k-1)}(R^{(k-1)}(\dots(R^{(k-1)}(k-1)+1)\dots)+1)+1,$$

where the function $R^{(k-1)}$ appears $s+t-2k+1$ times.

For $k=2$ this resolves to $R^{(2)}(s,t) \leq R^{(1)}(R^{(1)}(\dots(R^{(1)}(1)+1)\dots)+1)+1 = 2^{s+t-3}$, indicating

that the above proof is indeed the generalization of the argument in the proof of Theorem 2.2 from Week 1.
For $k = 3$ we obtain

$$R^{(3)}(s,t) \leq R^{(2)}(R^{(2)}(...(R^{(2)}(2)+1)...)+1)+1 \leq 2^{2\cdot R^{(2)}(...(R^{(2)}(2)+1)...)+1)-1}$$
$$\leq 2^{2^{2\cdot R^{(2)}(...(R^{(2)}(2)+1)...))}-1} \leq \cdots \leq 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}},$$

a tower of height $t + s - 3$.
For uniformity $k = 4$ things get completely out of hand: we have $t + s - 3$ 3-uniform Ramsey functions, each being a tower function, embedded inside one another ... To see a concrete example, let us note that the bound we get for $R^{(4)}(5,5)$, which is our upper bound on the Happy-Ending number $HE(5) = 9$, is a tower of 29 twos. For an upper bound on $17 = HE(6) \leq R^{(4)}(6,5)$ we have to take a tower of 2s of the height, which is a tower of 2s of height 29. This might question your intuitive understanding of the word finite...

Before proving the theorem again, let us analyse the proof given above and try to see why the bound became so incredibly large. We built our "right-monochromatic" sequence $v_1, v_2, ..., v_{s+t-2k+1}$ to this particular length so at the end we could use the 1-uniform Ramsey numbers for the sequence. For each new element of the sequence we had to apply the $(k-1)$-uniform Ramsey numbers. Our upper bound on the $(k-1)$-uniform Ramsey numbers are very large, their repeated application forces them to be embedded inside the arguments of the previous one and this causes the bound to become very-very-very-$\cdots$ -very large.
In contrast to this, the 1-uniform Ramsey bound, which we used only once at the end is rather small: only the sum of the two arguments (minus one. In fact this is not only a bound, but the exact value.) Erdős and Rado turned the proof idea on its head and tried to use the 1-uniform Ramsey numbers many times, but in exchange reduce the use of $(k-1)$-Ramsey numbers. They decided to build a much longer sequence by using the 1-dimensional Ramsey numbers in each round, so they have to use the $(k-1)$-uniform Ramsey numbers only once, at the end, for the sequence. This reduces the bound to a function we can actually write down. This function will still be large, but is relatively "close" to the truth.

Second proof of Theorem 2.2. Again we will apply induction on the uniformity k. The base case $k = 1$ was proved already.
Let $n \in \mathbb{N}$ be chosen large enough with respect to k, t, and s and let us be given an arbitrary two-coloring $c : \binom{[n]}{k} \to \{\texttt{red}, \texttt{blue}\}$ of the k-subsets of $[n]$. Our goal is to find either a monochromatic t-element subset $T \subseteq [n]$ in `red` (i.e. $c(Q) = \texttt{red}$ for every $Q \in \binom{T}{k}$) or a monochromatic s-element subset $T \subseteq [n]$ in `blue` (i.e. $c(Q) = \texttt{blue}$ for every $Q \in \binom{T}{k}$).
Our goal is to build a sequence $v_1, \ldots, v_\ell$ of vertices that is *right-$(k-1)$-neighborhood-monochromatic*. By this we mean that the color of a k-set contained in $\{v_1, \ldots, v_\ell\}$ should only depend on its $k-1$ smallest indices. Formally, by this we mean that there exists a coloring $c^* : \binom{[\ell-1]}{k-1} \to \{\texttt{red}, \texttt{blue}\}$ such that for every k-subset $T \subseteq \{v_1, \ldots, v_\ell\}$, we have $c(T) = c^*(T_{\min}(k-1))$, where we adopted the notation $T_{\min}(k-1) \in \binom{[\ell-1]}{k-1}$ for the set of the $k-1$ smallest indices j of vertices $v_j \in T$.

How long a sequence should we build? We observe that if we manage to build a right-$(k-1)$-neighborhood monochromatic sequence of length $\ell = R^{(k-1)}(t-1, s-1) + 1$, then we are done. The $(k-1)$-subsets of $[\ell - 1]$ are colored according to c^*. By the property of the Ramsey number $R^{(k-1)}(t-1, s-1)$, we find an index subset $I \subseteq [\ell - 1]$ of size $t - 1$ which is monochromatic `red` under c^* or an index subset $I \subseteq [\ell - 1]$ of size $s - 1$ which is monochromatic `blue` under c^*. Then we claim that the subset $T := \{v_i : i \in I\} \cup \{v_\ell\}$ is c-monochromatic in the same color as I is c^*-monochromatic in. And then of course it also has the required size (t in case of color `red` and s in case of color `blue`). To see this, let us take an arbitrary k-element subset $Q \subseteq T$. By the right-$(k-1)$-neighborhood monochromatic property of the coloring c^* we have that $c(Q) = c^*(Q_{\min}(k-1))$, where $Q_{\min}(k-1) \subseteq I$ is a $(k-1)$-element subset of the c^*-monochromatic index subset I.

So to complete the proof of our theorem we need to construct the right-$(k-1)$-neighborhood monochromatic sequence of the required length. Our plan is to construct a sequence $v_1, \dots v_i$ recursively. We will maintain a set N_i of vertices that are still "eligible" to be added to the sequence, that is, the c-color of any k-subset of vertices of $V_i := \{v_1, \dots v_i\} \cup N_i$ with at least $k-1$ vertices among $v_1, \dots, v_i$ indeed only depends on the $k-1$ smallest indices. We will pick the next vertex v_{i+1} arbitrarily from N_i and then reduce N_i to create N_{i+1} in order for the right-$(k-1)$-neighborhood monochromatic property also to hold for k-subsets involving the new vertex v_{i+1}.

To start let us select arbitrary vertices $v_1, \dots v_{k-2} \in [n]$ and set $N_{k-2} = [n] \setminus \{v_1, \dots, v_{k-2}\}$. Suppose that $i \geq k-2$ and we are given a sequence $v_1, \dots, v_i$ and a set N_i disjoint from it, such that there exists a coloring $c^* : \binom{[i]}{k-1} \to \{\texttt{red}, \texttt{blue}\}$ with the property that for every k-subset $T \subseteq \{v_1, \dots, v_i\} \cup N_i$ with $|T \cap \{v_1, \dots, v_i\}| \geq k-1$, we have $c(T) = c^*(T_{\min}(k-1))$. Note that such a sequence $v_1, \dots, v_i$ is always right-$(k-1)$-neighborhood monochromatic and that our initial choices vacuously satisfy the condition.

We choose the next vertex $v_{i+1} \in N_i$ arbitrarily. In order to designate $N_{i+1} \subseteq N_i \setminus \{v_{i+1}\}$, we define a function $\tilde{c} : N_i \setminus \{v_{i+1}\} \to \{\texttt{red}, \texttt{blue}\}^{\binom{i}{k-2}}$, such that the components of $\tilde{c}(w)$ for a vertex $w \in N_i \setminus \{v_{i+1}\}$ correspond to the $(k-2)$-subsets of $[i]$, and for the component corresponding to a subset $L \in \binom{[i]}{k-2}$ we have

$$\tilde{c}(w)_L := c(L \cup \{v_{i+1}, w\}).$$

This function is so defined that if we chose N_{i+1} to be the $\tilde{c}$-inverse image of any $\texttt{red}/\texttt{blue}$-vector, then we ensure that the desired property of the sequence $v_1, \dots, v_i$ and the function c^* extends to the $(k-1)$-element index subsets containing $i+1$. Indeed, choosing N_{i+1} to be the inverse image of the fixed $\texttt{red}/\texttt{blue}$-vector $\alpha \in \{\texttt{red}, \texttt{blue}\}^{\binom{i}{k-2}}$, we can extend c^* to $\binom{[i+1]}{k-1}$ as follows. The function c^* is already defined for sets in $\binom{[i]}{k-2}$, now choose an index set $I \in \binom{[i+1]}{k-2}$ that contains $i+1$ and define $c^*(I) := \alpha_{I \setminus \{i+1\}}$. To check that this definition is in line with what is desired from c^*, let us take any k-subset $Q \subseteq \{v_1, \dots, v_{i+1}\} \cup N_{i+1}$, with $v_{i+1} \in Q$ and having $|Q \cap \{v_1, \dots v_{i+1}\}| = k-1$. Then Q is of the form $Q = \{v_j : j \in J\} \cup \{v_{i+1}, w\}$, where $J \subseteq \binom{[i]}{k-2}$ and $w \in N_{i+1}$. In particular we have $Q_{\min}(k-1) = J \cup \{i+1\}$. By definition of $\tilde{c}$, and c^*, and since $w \in \tilde{c}^{-1}(\alpha)$, we obtain

$$c(Q) = \tilde{c}(w)_J = \alpha_J = c^*(J \cup \{i+1\}) = c^*(Q_{\min}(k-1)),$$

verifying the desired property of c^*.

To make N_{i+1} large we choose the largest possible $\tilde{c}$-inverse image. Hence N_{i+1} can be chosen so its size is at least the average size of an inverse image, that is

$$|N_{i+1}| \geq \left\lceil \frac{|N_i| - 1}{2^{\binom{i}{k-2}}} \right\rceil, \tag{1}$$

for every $i \geq k-2$.

Now let us estimate the size of n required in this proof. To make the sequence long enough, that is, to be able to choose the ℓth element of the sequence, we need that the set $N_{\ell-1}$ has at least one element. To ensure that $|N_{\ell-1}| \geq 1$ we use (1) repeatedly and choose $n = |N_{k-2}| + k - 2$ large enough. Namely, if $|N_{k-2}| \geq 2^{\binom{\ell}{k-2}}$ then we can choose the subsets $N_{k-2} \supseteq N_{k-1} \supseteq \cdots \supseteq N_{\ell-1}$, such that

$$2^{\binom{\ell-1}{k-2}} \leq |N_{k-2}| \leq 2^{\binom{k-2}{k-2}} |N_{k-1}| \leq 2^{\binom{k-2}{k-2}} \cdot 2^{\binom{k-1}{k-2}} |N_k| \leq \cdots \leq \prod_{j=k-2}^{\ell-2} 2^{\binom{j}{k-2}} |N_{\ell-1}| =$$

$$= 2^{\sum_{j=k-2}^{\ell-2} 2^{\binom{j}{k-2}}} \cdot |N_{\ell-1}| = 2^{\binom{\ell-1}{k-2}} \cdot |N_{\ell-1}|,$$

implying that $N_{\ell-1}$ is not empty and v_ℓ can be chosen.

In conclusion, the choice

$$n = 2^{\binom{\ell-1}{k-1}} + k - 2 = O\left(2^{\ell^{k-1}}\right)$$

with

$$\ell = R^{(k-1)}(t, s)$$

is sufficiently large for our selection process to go through and thus provides an upper bound on the Ramsey number $R^{(k)}(t, s)$. $\square$

By repeated application of this theorem we obtain a greatly improved upper bound compared to the first proof. We highlight this here by explicitely writing out the bound for the symmetric case $t = s$.

Corollary 2.3. *$R^{(k)}(t, t)$ is upper bounded by a tower function of t of height k.*

This upper bound is actually not that far from the truth. There is a construction of a `red`/`blue`-coloring of the k-sets without a monochromatic t-clique on a vertex set of size that is a tower function of t of height $k - 1$. To decide which height is the truth, even just for the 3-uniform Ramsey function, is worth a \$500 reward (by Erdős).