Canonical Ramsey, Two-colorable Hypergraphs

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1 The hypergraph Ramsey theorem

1.1 The Canonical Ramsey Theorem

In this section, we temporarily abandon our pursuit of the various bounds in "quantitative" Ramsey theory and return to the "qualitative" philosophical orgins of "complete disorder is impossible". The fundamental question of Ramsey theory is: given a classification (i.e., a coloring) of the elements of some structure, what sort of "order" can one necessarily find in it? We have seen many examples where a structure is colored with an arbitrary finite number of colors and we concluded the existence of a large "orderly" substructure (where by "orderly" we meant a substructure that is monochromatic).
An instance of this was the infinite Ramsey theorem. This can be generalized for hypergraphs and arbitrary finite number of colors.

Theorem 1.1 (HW). *For any positive integer r and any r-colouring of $\binom{\mathbb{N}}{k}$, there exists an infinite set $S \subseteq \mathbb{N}$ for which $\binom{S}{k}$ is monochromatic.*

In this subsection we will ask ourselves whether complete disorder would still be impossible if we colored our structire by *infinitely* many colors. We immediately realize that we must revise our notion of "orderly" substruture, as coloring each pair in $\binom{\mathbb{N}}{2}$ by a different color will not even leave us a monochromatic subset of size three![1]
In light of this example it seems necessary to include the situation when *all* pairs of elements of a set have distinct colors among orderly structures. This motivates the following definition.

Definition 1.2. *Given a colouring $c : \binom{\mathbb{N}}{2} \to C$, a set $S \subseteq \mathbb{N}$ is called* c-rainbow *if no two pairs of S have the same color.*

In the above coloring example the whole $\mathbb{N}$ is a rainbow set. Is this enough for the concept of "orderly"? Are we always guaranteed to find either an infinite monochromatic set or an infinite rainbow set? The answer is still no. To see this, simply colour each pair $\{i, j\}$ with its minimal element $\min\{i, j\}$. In this coloring we still do not find a monochromatic set of size three, but neither find a rainbow set of size three. This example motivates the following definition.

Definition 1.3. *Given a colouring $c : \binom{\mathbb{N}}{2} \to C$, a set $S \subseteq \mathbb{N}$ is called* c-left-injective *if there is an injective map $c^* : \mathbb{N} \to C$, such that $c(ij) = c^*(\min\{i, j\})$.*

The name *left-injective* subset originates in its property that the colour of an edge is uniquely determined by its *left* endpoint. Note that the a left-injective subset forms a right-monochromatic sequence (from the last subsections).
Of course there is nothing special about the minimum, we could also define a coloring of $\binom{\mathbb{N}}{2}$ by setting the color of every edge to be the maximum of its endpoints. Then there is no monochromatic, no rainbox, and no right-injective set of size three. Hence analogously we define the notion of *right-injective* colouring.

[1] For notational simplicity we restrict ourselves further to the 2-uniform case; analogous results hold for arbitrary uniformity k.

Definition 1.4. *Given a colouring* $c : \binom{\mathbb{N}}{2} \to C$*, a set* $S \subseteq \mathbb{N}$ *is called* c-right-injective *if there is an injective map* $c^* : \mathbb{N} \to C$*, such that* $c(ij) = c^*(\max\{i, j\})$.

Surprisingly, it is not only necessary but also sufficient that we extend our notion of "orderly" subset to include these four cases: one of them will occur! This is stated in the next Canonical Ramsey Theorem.

Theorem 1.5 (Erdős-Rado, 1950)**.** *Let* $c : \binom{\mathbb{N}}{2} \to C$ *be a coloring. Then there is some infinite set* $S \subseteq \mathbb{N}$ *such that either*

i) S *is c-monochromatic, or*

ii) S *is c-rainbow, or*

iii) S *is c-left-injective, or*

iv) S *is c-right-injective.*

Remarks

- This is a strengthening of Ramsey's Theorem for finitely many colors from the Homework. Indeed, when the number of colors used is finite, then options (*ii*)-(*iv*) are impossible.)
- The colorings appearing on these four types of sets are called the *canonical* colorings. In case (*ii*) the color of an edge is determined injectively by both enpoints, in case (*iii*) it is determined by the left endpoint, in case (*iv*) it is determined by the right endpoint, and in case (*i*) it is just determined (by no endpoint). The theorem states that every colouring contains an infinite canonically coloured clique.

Proof. The idea of the proof, just like in the approach to the Happy-Ending Problem, is to try to use local informations to deduce something for the global structure. Since we will need to compare the colors on *pairs* of edges, we should be interested in the coloring of 4-element subsets. One of the main questions is how to reduce the number of colors to finite, so that we are able to use the 4-uniform Ramsey Theorem. To this we will colour the 4-subsets of $\mathbb{N}$ such that we encode the information about the color pattern on the edges between these four integers. This coloring will use only finitely many colors since we will only be interested in the colour pattern i.e. keeping track of which edges have the same colour and which do not, but we will **not** care exactly which particular colors we use to create this pattern. It might sound surprising at first that this information, the color pattern on 4-element sets, is sufficient to deduce the existence of an infinite set with a canonical coloring.

Formally, let us define a coloring $\hat{c} : \binom{\mathbb{N}}{4} \to \mathcal{B}\left(\binom{[4]}{2}\right)$ where $\mathcal{B}\left(\binom{[4]}{2}\right)$ is the set of all set partitions[2] of the six-element set $\binom{[4]}{2}$. The value $\hat{c}(\{i_1, i_2, i_3, i_4\})$ for some 4-subset $\{i_1, i_2, i_3, i_4\}$ with $i_1 < i_2 < i_3 < i_4$ is just the set partition that is induced by the inverse images of c on the set $\binom{\{i_1,i_2,i_3,i_4\}}{2}$ and hence in turn on the set $\binom{[4]}{2}$.

For example, the value of $\hat{c}$,

- for a rainbow 4-set is $\{\{12\}, \{13\}, \{14\}, \{23\}, \{24\}, \{34\}\}$,
- for a monochromatic 4-set is $\{\{12, 13, 14, 23, 24, 34\}\}$,
- for a left-injective subset is $\{\{12, 13, 14\}, \{23, 24\}, \{34\}\}$,
- for a right-injective subset is $\{\{12\}, \{13, 23\}, \{14, 24, 34\}\}$.

[2]The cardinality of $\mathcal{B}\left(\binom{[4]}{2}\right)$ is the sixth Bell number $B_6 = 203$, which is the sum of the Stirling numbers $S(6, k)$of the second kind with the summation running till $k = 6$.

We use the 4-uniform Ramsey Theorem for 203 colors and find an infinite $\hat{c}$-monochromatic subset $S = \{s_1 < s_2 < \cdots < s_i < \cdots\} \subseteq \mathbb{N}$. In other words there is a set partition $\mathbf{p} \in \mathcal{B}\left(\binom{[4]}{2}\right)$ such that for every 4-subset $T = \{i_1 < i_2 < i_3 < i_4\} \subseteq S$ we have $c(i_u i_v) = c(i_w i_z)$ for some $uv, wz \in \binom{[4]}{2}$ if and only if uv and wz are in the same class of the set partition $\mathbf{p}$. Now we have a little case distinction based on how $\mathbf{p}$ looks like.
Case 1. $\mathbf{p} = \{\{12\}, \{13\}, \{14\}, \{23\}, \{24\}, \{34\}\}$. In this case the whole S is rainbow. Indeed, for any two edges s_1s_2 and s_3s_4, there exists a 4-element subset T containing both of these edges. Since $\hat{c}(T) = \mathbf{p}$, all edges, in particular also s_1s_2 and s_3s_4 have distinct c-colors.
Case 2. There is a partition class of $\mathbf{p}$ with at least two pairs.
We will show that c induces a canonical coloring on $S_{\texttt{even}} = \{s_{2i} : i \in \mathbb{N}\}$.
Case 2a. There are two pairs that are disjoint and are contained in the same partition class of $\mathbf{p}$. If 12 and 34 are in the same class then S is an infinite c-monochromatic subset, as the color of any two edges s_xs_y and s_us_v are equal since they are both equal to the color of the edge s_as_{a+1}, say with $a = \max\{x, y, u, v\} + 1$.
If 14 and 23 are in the same class then $S \setminus \{s_1\}$ is an infinite c-monochromatic subset.
If 13 and 24 are in the same class, then $S_{\texttt{even}} = \{s_{2i} : i \in \mathbb{N}\}$ is an infinite c-monochromatic subset.
Case 2b. Every two pairs that are disjoint are in different partition classes of $\mathbf{p}$.
Consequently there must be two pairs xz and $yz \in \binom{[4]}{2}$ that are in the same partition class of $\mathbf{p}$. Without loss of generality $x < y$.
If $x < z < y$, then $S_{\texttt{even}}$ is c-monochromatic.
If $z < y < x$, then $S_{\texttt{even}}$ is c-left-injective.
If $y < x < z$, then $S_{\texttt{even}}$ is c-right-injective. □

Remark. The Canonical Ramsey Theorem extends to the k-uniform setting. There we will have to admit 2^k canonical colourings. (Homework)

2 Ramsey theory and the chromatic number of hypergraphs

The first superficial but usual line of confusion with the r-colorings of the edges of the complete graph in graph Ramsey theory is that they have something to do with proper colorings or proper edge colorings as one learns these useful concepts before. And they don't. Now we describe a point of view which shows how it is in fact closely related to proper coloring of hypergraphs.
In the definition of proper coloring of a graph we require that the colors of neighboring vertices are different. This can be generalized in various ways to hypergraphs; the concept we will be dealing with forbids hyperedges that are monochromatic.

Definition 2.1. *Given a hypergraph $\mathcal{H}$ on vertex set V, a function $c : V \to [r]$ is called* proper r-coloring *of $\mathcal{H}$ if every edge $e \in \mathcal{H}$ contains vertices with different colors. The* chromatic number *$\chi(\mathcal{H})$ of the hypergraph is the smallest r for which there exists a proper r-coloring of $\mathcal{H}$.*

It turns out that all our symmetric Ramsey statements can be naturally formulated as statements about the chromatic number of a special hypergraph. To this end let us define for positive integers $n \geq t \geq k$ the Ramsey hypergraph $\mathcal{R}^{(k)}_{n,t}$ as the $\binom{t}{k}$-uniform hypergraph on the vertex set $\binom{[n]}{k}$ where

$$\mathcal{R}^{(k)}_{n,t} = \left\{ \binom{T}{k} : T \subseteq [n], |T| = t \right\}.$$

That is, for each vertex subset of size t we define a hyperedge in $\mathcal{R}^{(k)}_{n,t}$, containing all the k-subsets of the clique induced by T. Hence the number of hyperedges of $\mathcal{R}^{(k)}_{n,t}$ is $\binom{n}{t}$.
In Ramsey theory we were studying the chromatic number of the hypergraph $\mathcal{R}^{(k)}_{n,t}$ for various values of the parameters n, t, k. Indeed, in the most classical of all Ramsey problems, a coloring of the *edges of K_n* does not contain a monochromatic K_k if and only if the corresponding *vertex*

coloring of the hypergraph $\mathcal{R}^{(2)}_{n,t}$ is proper. In other words $R(t,t) \le n$ if and only if $\chi\left(\mathcal{R}^{(2)}_{n,t}\right) > 2$, that is we are looking for the smallest integer n when the hypergraph $\mathcal{R}^{(2)}_{n,t}$ is *not* 2-colorable.
To formulate the infinite Ramsey theorem in terms of chromatic number, one can introduce for any positive integers k the infinite Ramsey hypergraph $\mathcal{R}^{(k)}_{\infty}$ on vertex set $\binom{\mathbb{N}}{k}$ where

$$\mathcal{R}^{(k)}_{\infty} = \left\{ \binom{T}{k} : T \subseteq \mathbb{N}, |T| = \infty \right\}.$$

The infinite Ramsey theorem then simply states that $\chi\left(\mathcal{R}^{(k)}_{\infty}\right) = \infty$ (since no r-colouring is proper, for any positive integer r).

2.1 Property B

The above is just one motivation us to study 2-colorability of hypergraphs. The manifestation of the concept for graphs is extremely useful: bipartite graphs are known to model many scenarios and their theory is well-developed. They are easy to recognize. This is not the case for hyoergraphs: deciding the 2-colorability of given hypergraph is an NP-hard problem, already for uniformity 3. In case of an NP-hard decision problem one does what one can, find meaningful conditions guaranteeing that the property is satisfied. Next we will concern ourselves with finding sufficient conditions for the 2-colorability of a hypergraph in terms of its number of edges. If a k-uniform hypergraph, with $k \ge 2$, has only one edge (or just a few edges) then it is for sure easily 2-colorable. We will be after determining how few is this few as a function of the uniformity k of the hypergraph.
To this end let us define $m_B(k)$ to be the smallest integer m such that there exists a non-2-colorable k-graph with m edges. The letter B signals the traditional name for the 2-colorability property of hypergraphs.[3]
To internalize this function, let us think over what follows and what does *not* follow from $m_B(k) = m$.

- $\Rightarrow$ **there is** a k-graph with m edges that is not 2-colorable.
- $\Rightarrow$ **every** k-graph with $m-1$ edges is 2-colorable.
- $\not\Rightarrow$ every k-graph with m edges is not 2-colorable. (eg. disjoint union of arbitrarily many k-edges is always 2-colorable)

Iterating the above theme let us spell out what needs to be done in order to prove bounds on $m_B(k)$.

- To have m as an **upper bound**, we need to give a construction (or prove the existence) of a k-graph with m edges, such that every 2-coloring of it contains a monochromatic edge.
- To have m as a **lower bound**, we need to properly 2-color every k-graph with $m-1$ edges.

Let us practice with the case $k = 2$. What is $m_B(2)$? For an upper bound we can give K_3, which is a non-2-colorable 2-graph with three edges. This shows that $m_B(2) \le 3$. For a lower bound we note that every graph with two edges is 2-colorable. So $m_B(2) = 3$.
Let us generalize the above complete graph construction and give *some* upper bound for every k. For what values of n will the complete k-graph $\mathcal{K}^{(k)}_n = \binom{[n]}{k}$ be non-2-colorable? For $n \le 2k-2$, it is 2-colorable since one can color at most $k-1$ vertices `red` and at most $k-1$ vertices `blue` and not have any monochromatic k-edge.
For $n = 2k-1$ however, no matter how we color, by the PP one of the color classes will have size at least $\lceil \frac{2k-1}{2} \rceil = k$. This gives a monochromatic k-edge and shows that the coloring was not proper. The number of edges in this hypergraph is $\binom{2k-1}{k}$, which gives an upper bound $O\left(\frac{4^k}{\sqrt{k}}\right)$.

[3]Property B was introduced by Felix Bernstein in 1908

Do we really need all the edges of $\mathcal{K}_{2k-1}^{(k)}$ for it to be non-2-colorable? Yes, we do! If some edge $e \in \mathcal{K}_n^{(k)}$ was missing from the hypergraph, then we could color exactly these k vertices by `red` and the remaining $k-1$ `blue` and thus produce a proper 2-coloring.
So the complete k-graph on $2k-1$ vertices is a minimal non-2-colorable k-graph on the fewest possible number of vertices. Could this be the one also with the fewest number of edges? It is for $k = 2$.
To investigate this further we try to give some lower bound. For that we will need to give a proper 2-coloring of *any* k-graphs with "few" edges. Considering that we do not know *anything* about the structure of k-graph to be 2-colored[4], this appears to be a daunting task. In such cases we get used to turning to randomness for the rescue.

Theorem 2.2 (Erdős, 1963)**.** *For all $k \geq 2$, $m_B(k) > 2^{k-1}$.*

Proof. Let $\mathcal{H}$ be a k-graph with $m \leq 2^{k-1}$ edges. We produce a random coloring c_{rand} by coloring each vertex of $\mathcal{H}$ uniformly at random by `red` and `blue` with these choice being mutually independent. We will draw our conclusions from the expectation of the random variable X counting the number of monochromatic edges of $\mathcal{H}$.
As always, we try to break X into a sum of simple indicator variables. To this end, we introduce a "bad event" E_e for each edge $e \in \mathcal{H}$, representing that e is monochromatic under c_{rand}. We define X_e to be the indicator random variable of E_e, so we can write $X = \sum_{e \in \mathcal{H}} X_e$. The probability of E_e and hence the expectation of X_e is $\frac{2}{2^k}$, because each of the 2^k coloring of e is equally likely and exactly 2 of them (the constant `red` and the constant `blue`) produce a monochromatic e. Then, using the linearity of expectation,

$$\mathbb{E}[X] = \sum_{e \in \mathcal{H}} \mathbb{E}[X_e] = \sum_{e \in \mathcal{H}} \frac{2}{2^k} = \frac{|\mathcal{H}|}{2^{k-1}}.$$

If the expectation of X is less than 1, that is if $|\mathcal{H}| < 2^{k-1}$, then there must be a coloring c such that $X(c) < 1$. Since X takes only integer values, we must also have that $X(c) = 0$, so c creates no monochromatic edge. Now if $|\mathcal{H}| = 2^{k-1}$, that is if $\mathbb{E}[X] = 1$, and there is *no* coloring c with $X(c) = 0$, then for every single coloring there must be exactly one monochromatic set. This is clearly not the case, as the all `red` coloring for example creates $|\mathcal{H}| > 1$ monochromatic edges. Consequnetly there must again be a coloring c, with $X(c)$. □

Now that we have both an upper and lower bound of exponential order, 4^k and 2^k, respectively, we revisit our simple upper bound. Recall that the complete k-graph on $2k-1$ was minimal with respect to being non-2-colorable. Our plan is to use more vertices but much less edges and guarantee non-2-colorability via random selection of the edges. This will close the gap between upper and lower bound in terms of the base of the exponential.

Theorem 2.3 (Erdős, 1964)**.**

$$m_B(k) = O(k^2 2^k).$$

Proof. We choose a vertex set V of size n, to be determined later in the proof. We pick $m = O(k^2 2^k)$ edges uniformly at random, with replacement, from all k-sets in V. We do this so the choices are mutually independent. This way we create a random k-graph $\mathcal{H}_{rand}$ with *at most* m edges. (It is not necessarily equal, because we pick with replacement, so it could happen that some edge is picked more than once, in which case we just ignore this choice to make our hypergraph simple.)
Our goal is to show that no 2-coloring is proper. To this end we will show the slightly stronger (and certainly more easy to analyse) statement that every half of the vertex set V contains a hyperedge. This implies that no 2-coloring is proper, since one of the two color classes will be of size at least $\frac{|V|}{2}$ and hence contain an edge.

[4] You can really imagine that a really evil spirit hands it to you and you still *must* properly 2-color it.

We introduce a "bad event" E_S for each subset $S \in \binom{V}{n/2}$, representing that S does *not* contain any of the m random hyperedges we picked independently. What is the probability that a uniformly random k-set is not contained in S? The number of choices is $\binom{n}{k}$ and among these $\binom{n/2}{k}$ are contained in S. So

$$\mathbb{P}[E_S] = \left(\frac{\binom{n}{k} - \binom{n/2}{k}}{\binom{n}{k}} \right)^m,$$

since the choices of the m hyperedges are mutually independent. To estimate $\mathbb{P}[E_S]$ from above we first consider

$$\frac{\binom{n/2}{k}}{\binom{n}{k}} = \prod_{j=0}^{k-1} \frac{\frac{n}{2} - j}{n-j} = \prod_{j=0}^{k-1} \frac{1}{2} \cdot \left(1 - \frac{j}{n-j}\right) \geq \frac{1}{2^k} \left(1 - \frac{k}{n-k}\right)^k \geq \frac{1}{2^k} \left(1 - \frac{k^2}{n-k}\right).$$

The last inequality is an application of Bernoulli's inequality[5] stating $(1+x)^k \geq 1 + kx$ for every real number $k \geq 1$ and real number $x \geq -1$. Observe that this bound indicates that we better choose our n to be more than k^2+k, otherwise our lower bound on this probability is negative—not particularly useful.

We estimate now the existence of a bad event through the union bound:

$$\mathbb{P}\left[\bigcup_{S \in \binom{V}{n/2}} E_S \right] \leq \sum_{S \in \binom{V}{n/2}} \mathbb{P}[E_S] \leq \binom{n}{n/2} \cdot \left(1 - \frac{1}{2^k}\left(1 - \frac{k^2}{n-k}\right)\right)^m \leq 2^n \cdot e^{-m\frac{1}{2^k}\left(1-\frac{k^2}{n-k}\right)}$$
$$< e^{n - \frac{m}{2^k}\left(1 - \frac{k^2}{n-k}\right)}$$

To make the probabilty of the union of bad events less than 1, we make sure that the exponent of the final estimate is less than 0. This happens exactly when

$$m > n \cdot 2^k \cdot \frac{1}{1 - \frac{k^2}{n-k}}.$$

Choosing $n = 2k^2 + k$ we get that the probability that every half of V contains at least one of

$$m = (2k^2 + k) \cdot 2^k \cdot \frac{1}{1 - \frac{k^2}{2k^2+k-k}} = (2k^2 + 2) \cdot 2^k \cdot 2 = O(k^2 2^k)$$

uniformly random k-sets is positive. Consequently there *is* a way to select m k-edges like that, which edges constitute a non-2-colorable k-graph.

□

Remarks. More careful optimization in the final estimates gives $m_B(k) \leq (1 + o(1))\frac{e \ln 2}{4} k^2 2^k$. We now have

$$\Omega(2^k) = m_B(k) = O(k^2 2^k),$$

so we determined the exponential growth rate. The determination of the correct polynomial factor is still outstanding. Next we will be concerned with that. We will see a very clever, and very different probabilistic argument to improve the lower bound.

[5] The special case $(1-x)^k \geq 1 - kx$, for $k \in \mathbb{N}$ and $x \in [0,1]$, can also be interpreted as the Bonferroni inequality which essentially amounts to the union bound. It can be proven by interpreting x as the probability of k mutually independent events.