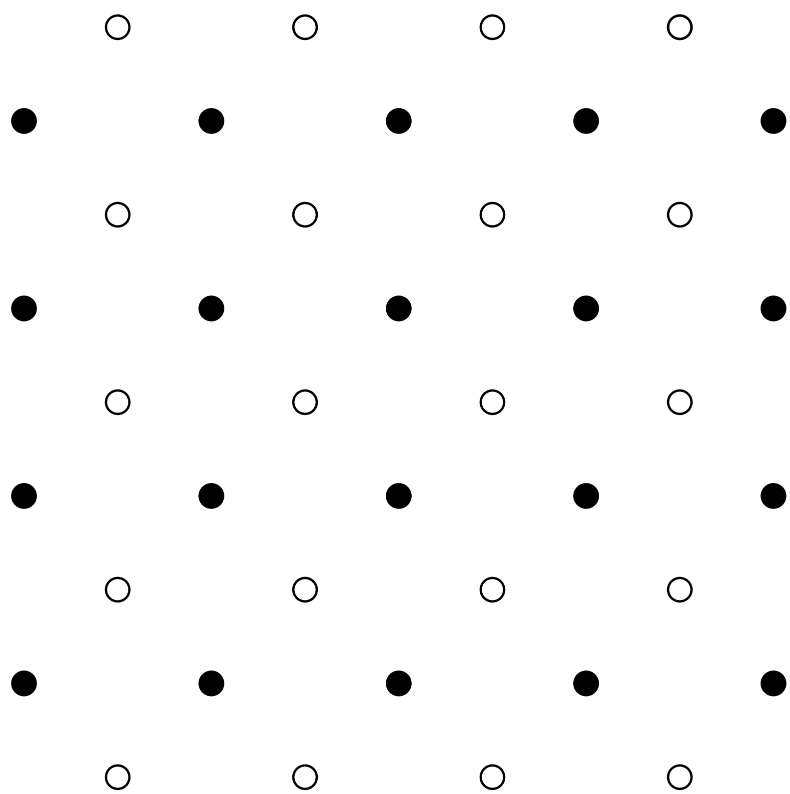
Bridg-it* by David Gale



Who wins in Bridg-it?

Theorem. Player 2 does not have a winning strategy.

Proof. **Strategy Stealing:**

Suppose Player 2 has a winning strategy S. Then here is a winning strategy for Player 1:

Start with an arbitrary edge e_1, then pretend to be Player 2 and play according to S (Possible since playground is symmetric), forgetting about m for this purpose. If S calls for playing m, then pretend you have just played it, and select another arbitrary edge e_2 to play, which you will ignore for the purposes of your strategy. Etc...

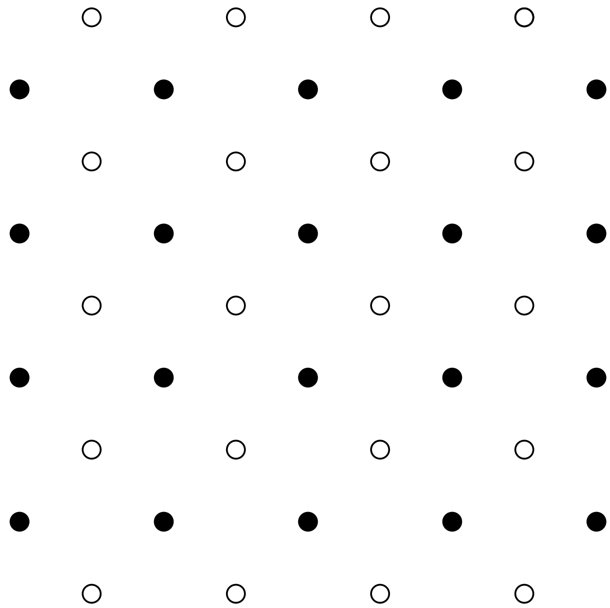
Since you play according to a winning strategy, you win! But if Player 2 was also playing according to S, then he also wins! $\Rightarrow$ contradiction, since both cannot win. $\square$

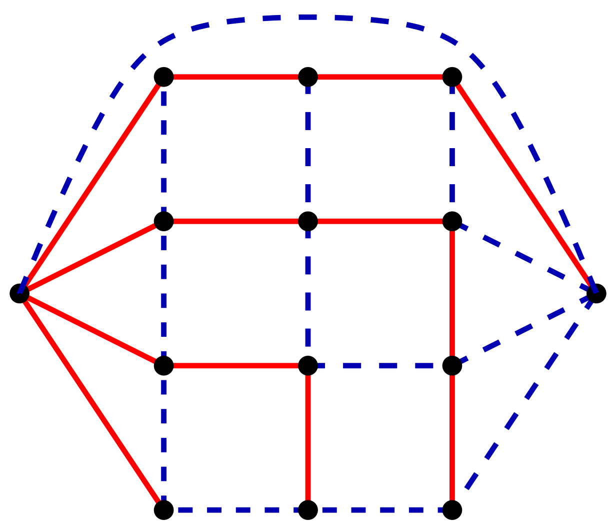
Since there is **no** final position which is a draw (HW), Player 1 **must have** a winning strategy!

Good to know, but HOW TO WIN???*

*In HEX, similary, we do know that Player 1 has a winning strategy, but NO explicit strategy is known! ($\rightsquigarrow$ still fun game!)

An explicit strategy via spanning trees





The game of “Connectivity”

A positional game is played by two players, Maker and Breaker, who alternately take edges of a base graph G. Maker uses a permanent marker, Breaker uses an eraser. Maker wins the positional game “Connectivity” if by the end he occupies a connected subgraph of G. Otherwise Breaker wins.

Theorem. (Lehman, 1964) Suppose Breaker starts the game. If G contains two edge-disjoint spanning tree, then Maker has an explicit winning strategy in “Connectivity”.

Proof. Maker maintains two spanning trees T_1 and T_2, such that after each full round,

Maker) (i) $E(T_1) \cap E(T_2)$ consists of the edges claimed by Maker,

(ii) $E(T_1) \triangle E(T_2)$ contains only unclaimed edges.

Remark. The other direction of the Theorem is also true.

The tool for Player 1. (i.e. Maker)

Proposition. If T and T' are spanning trees of a connected graph G and $e \in E(T) \setminus E(T')$, then **there is** an edge $e' \in E(T') \setminus E(T)$, such that $T - e + e'$ is a spanning tree of G.

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