Algorithmic Combinatorics Wi 2018–19
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 10

Due date: 16:15, 15th January

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 If you forget the Proposal Algorithm for the stable matching problem, you might try to find a stable matching by using the Hungarian Algorithm instead. Given men $\{M_1, M_2, \dots, M_n\}$ and women $\{W_1, W_2, \dots, W_n\}$, each with their own preference lists of the members of the opposite gender, a natural edge-weighting would be $\omega(\{M_i, W_j\}) = 2n-k-\ell$, where M_i is the kth man on W_j's list, and W_j is the ℓth woman on M_i's list.

Show that for every (large enough, if needed) n, there are preference lists such that no maximum-weight matching is a stable matching.

Exercise 2 A natural analogue of the stable matching problem is the *stable roommate problem* where we have a set of $2n$ people, each with a preference order for the rest of the people, who want to partition themselves into pairs (roommates) in a stable way, i.e., there shouldn't be any pair of people who both prefer each other over their assigned roommates. Prove that a stable matching can fail to exist in this problem.

[Hint (to be read backwards): .elpoep ruof htiw ti yrT]

Exercise 3 A professor is computing the final homework grades for the 30 students, $S_1, S_2, \dots, S_{30}$, in her course. There were a total of 12 exercise sheets, $E_1, E_2, \dots, E_{12}$, each graded out of 20 points, and for $1 \leq i \leq 30$ and $1 \leq j \leq 12$, student S_i scored $p_{i,j}$ points on exercise sheet E_j. Each student needs to get an average score of 12 points to pass.

However, the professor has a trick up her sleeve — she never said that the assignments would carry equal weight. She is free to choose non-negative weights w_j, $1 \leq j \leq 12$, that will be used to average the students' grades. There is a subset $\mathcal{L} \subseteq \{S_1, S_2, \dots, S_{30}\}$ of the students that she likes, and she wants to make sure they will pass. On the other hand, there is a disjoint subset $\mathcal{D} \subseteq \{S_1, S_2, \dots, S_{30}\}$ of students she dislikes, and she wants to ensure they will fail.

Set up, with explanation, a linear program to help her find suitable weights w_j.

Exercise 4 We are given a convex n-gon C in $\mathbb{R}^2$, and we want to find the largest disk that fits inside C. Set up, with explanation, a linear program to find the centre and the radius of this largest disk.

[Hint (to be read backwards): $?b + xa = y$ enil a rof (q, p) tniop a fo ecnatsid eht si tahW]

Exercise 5 The decision version of an integer linear programming (ILP) asks for the existence of a feasible integer solution satisfying the given constraints. Use the fact that 3-SAT is NP-complete to prove that the decision version of ILP is NP-complete.

Exercise 6 Let $G = (V, E)$ be a graph with $V = [n]$. Consider the following integer linear program on variables x_i and y_{jk} for $i, j, k = 1, \ldots, n$.

$$\begin{array}{lrr} \text{min} & \sum_{i=1}^{n} x_i & \\ \text{subject to} & \sum_{k=1}^{n} y_{jk} = 1 & j = 1, \ldots, n \\ & y_{ik} + y_{jk} \leq 1 & (i, j) \in E, k = 1, \ldots, n \\ & y_{jk} - x_k \leq 0 & j, k = 1, \ldots, n \\ & x_i, y_{jk} \in \{0, 1\} & i, j, k = 1, \ldots, n \end{array}$$

For which graph parameter f, is $f(G)$ equal to the optimal value of the integer program above?