Algorithmic Combinatorics Wi 2018–19
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Exercise Sheet 11

Due date: 16:15, 29th January

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 Consider the linear program given by maximising $\vec{c}^{\,T}\vec{x}$, subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$, where $\vec{c} = (1,1,1,1)^T$, $\vec{b} = (5,4)^T$, and

$$A = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 2 & 2 & 1 & -2 \end{pmatrix}.$$

Determine which pairs of columns form bases, which of those bases are feasible, and solve the linear program.

Exercise 2 Use the simplex algorithm to solve the following linear program.

$$\begin{array}{lrcrl} \text{maximise} & x_1 & + & 2x_2 & \\ \text{subject to} & -2x_1 & + & 2x_2 & \geq 2, \\ & 2x_1 & + & x_2 & \leq 2, \\ & & & x_1 & \in \mathbb{R}, x_2 \geq 0. \end{array}$$

Exercise 3 Dantzig originally suggested that the entering variable be the one with the largest coefficient r_i. By considering the following linear program, show that the simplex algorithm can indeed cycle with this pivot rule.

$$\begin{array}{lrcrcrcrl} \text{maximise} & 46x_1 & + & 43x_2 & - & 271x_3 & - & 8x_4 & \\ \text{subject to} & 2x_1 & + & x_2 & - & 7x_3 & - & x_4 & \leq 0, \\ & -39x_1 & - & 7x_2 & + & 39x_3 & + & 2x_4 & \leq 0, \\ & & & & & & & \vec{x} & \geq \vec{0}. \end{array}$$

Exercise 4 Consider the linear program

$$\begin{array}{ll} \text{maximise} & \vec{c}^{\,T}\vec{x} \\ \text{subect to} & A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}. \end{array}$$

(a) A set $X \subseteq \mathbb{R}^n$ is said to be *convex* if whenever $\vec{x}, \vec{y} \in X$ and $\alpha \in [0,1]$, we have $\alpha\vec{x} + (1-\alpha)\vec{y} \in X$ as well. Show that the set of feasible solutions to our linear program is convex.

(b) As the intersection of finitely many half-spaces, the feasible set is in fact a convex *polyhedron* in $\mathbb{R}^n$. A *vertex* of a polyhedron P is a point $\vec{v} \in P$ such that there is some vector $\vec{c} \in \mathbb{R}^n$ with $\vec{c}^{\,T}\vec{v} > \vec{c}^{\,T}\vec{y}$ for all $\vec{y} \in P \setminus \{\vec{v}\}$. Prove that $\vec{x}$ is a vertex of the feasible set for our linear program if and only if it is a basic feasible solution.

Exercise 5 While running the simplex algorithm we arrive at a feasible basis B with simplex tableau $\mathcal{T}(B)$

$$\begin{aligned} \vec{x}_B &= \vec{p} + Q\vec{x}_N \\ z &= z_0 + \vec{r}^{\,T}\vec{x}_N \end{aligned}$$

has parameters $\vec{p} = (p_i)_{i \in B} \in \mathbb{R}^m$, $Q = (q_{i,j})_{i \in B, j \in N} \in \mathbb{R}^{m \times (n-m)}$, $\vec{r} = (r_j)_{j \in N} \in \mathbb{R}^{n-m}$ and $z_0 \in \mathbb{R}$. Suppose that this solution is not optimal, and we pivot to a basis $B' = B \setminus \{u\} \cup \{v\}$. Let $\mathcal{T}(B')$ be the new tableau, with parameters $\vec{p}', Q', \vec{r}'$ and z_0'. Find formulae for $p_i', q_{i,j}', r_i'$ and z_0' in terms of $p_i, q_{i,j}, r_i$ and z_0.

Exercise 6 The following is an example due to Klee and Minty that shows that Dantzig's pivot rule of largest coefficient can lead to exponentially many steps in the Simplex algorithm.

$$\begin{array}{lll} \text{maximise} & 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2x_{n-1} + x_n & \\ \text{subect to} & x_1 & \leq 5 \\ & 4x_1 + x_2 & \leq 25 \\ & 8x_1 + 4x_2 + x_3 & \leq 125 \\ & \vdots & \vdots \\ & 2^n x_1 + 2^{n-1}x_2 + \cdots + 4x_{n-1} + x_n & \leq 5^n \\ & \vec{x} \geq 0. & \end{array}$$

(1) Prove that the optimum solution of this LP is at $(0, \ldots, 0, 5^n)$.

(2) For $n = 2, 3$, perform the simplex algorithm on this LP using Dantzig's pivot rule, starting at $\vec{x} = 0$.

(3) (Bonus) Prove that for all n, the simplex algorithm using Dantzig's pivot rule takes 2^n steps, starting at $\vec{x} = 0$.