Algorithmic Combinatorics — Wi 2018–19

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Exercise Sheet 12

Due date: 16:15, 5th February

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 Consider the following linear program.

$$\begin{array}{lrcrcrcl} \text{min} & 4x_1 & + & 100x_2 & + & 120x_3 & & \\ \text{subject to} & 15x_1 & - & x_2 & - & 3x_3 & \leq & 10, \\ & -x_1 & - & x_2 & - & x_3 & \leq & -1, \\ & & & & & \vec{x} & \geq & \vec{0}. \end{array}$$

Find its dual, and use it to find the optimal value of the original program.

Exercise 2 The Duality Theorem states that when considering the feasibility and boundedness of the primal and dual programs, at most four of nine cases can occur. Give, with justification, (small) linear programs and their duals to show that each of these four cases does actually occur.

Exercise 3 In the lecture we described the following general recipe to convert a linear program to its dual. Say we have a linear program where we are maximizing $\vec{c}^T\vec{x}$ with the i-th constraint as

$$a_{i1}x_1 + \cdots + a_{in}x_n \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} b_i,$$

$i = 1, \ldots, m$. Then the dual linear program has variables $y_1, \ldots, y_m$, where y_i corresponds to C_i and satisfies

$$\left\{ \begin{array}{c} y_i \geq 0 \\ y_i \leq 0 \\ y_i \in \mathbb{R} \end{array} \right\} \text{ if we have } \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} \text{ in } C_i.$$

The constraints of the dual linear program are $Q_1, \ldots, Q_n$ where Q_j corresponds to the variable x_j and reads

$$a_{1j}y_1 + \cdots + a_{mj}y_m \left\{ \begin{array}{c} \geq \\ \leq \\ = \end{array} \right\} c_j \text{ if } x_j \text{ satisfies } \left\{ \begin{array}{c} x_j \geq 0 \\ x_j \leq 0 \\ x_j \in \mathbb{R} \end{array} \right\}.$$

Using the Duality Theorem we proved in the lecture, prove that the first linear program is feasible and bounded if and only if the second one is, and in that case they have the same optimal value.

Exercise 4 Recall that a matrix A is called totally unimodular if every square submatrix of it has determinant 0, -1 or 1. In the lectures we showed that for an LP

$$\begin{array}{ll} \text{maximise} & \vec{c}^{\,T}\vec{x} \\ \text{subect to} & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq \vec{0} \end{array}$$

where $\vec{b} \in \mathbb{Z}^n$, if there is an optimal solution then there is an obtimal basic feasible solution with integer coordinates. Prove that in this case *every* basic feasible solution (irrespective of being optimal or not) has integer coordinates.

Exercise 5 Recall that in a network (D, s, t, c), where $D = (V, E)$ is a directed graph, s and t are the designated source and sink vertices, and $c : E \to \mathbb{R}_{\geq 0}$ is the edge capacity function, a feasible flow $f : E \to \mathbb{R}$ is a function such that $f^+(v) = f^-(v)$ for all $v \in V \setminus \{s, t\}$ and $0 \leq f(e) \leq c(e)$ for all $e \in E$. The max-flow is defined as the maximum value of $f^-(t) - f^+(t)$ over all feasible flows f. The min-cut is defined as the minimum capacity of an edge cut $[S, \overline{S}]$ with $s \in S$ and $t \in \overline{S}$.

(1) Formulate the problems of finding a max-flow and a min-cut as dual linear programs of each other and thus deduce the max-flow min-cut theorem.

(2) Prove that the matrix of this linear program is totally unimodular and deduce the integrality theorem which states that if $c : E \to \mathbb{Z}$ then the max-flow is an integer.

Exercise 6 Given an arbitrary linear program (P), construct another linear program (P)$'$, such that (P) has an optimal solution if and only if (P)$'$ has a feasible solution.

Exercise 7 (**Bonus 10 pnts**) Show that the following three statements are equivalent (you do not have to prove any of them).

(i) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ has a non-negative solution if and only if every $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^{\,T}A \geq \vec{0}^{\,T}$ also satisfies $\vec{y}^{\,T}\vec{b} \geq 0$.

(ii) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} \leq \vec{b}$ has a non-negative solution if and only if every non-negative $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^T A \geq \vec{0}^T$ also satisfies $\vec{y}^T\vec{b} \geq 0$.

(iii) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} \leq \vec{b}$ has a solution if and only if every non-negative $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^T A = \vec{0}^T$ also satisfies $\vec{y}^T\vec{b} \geq 0$.

Use any of these three statements to give an alternative proof of the Duality Theorem that avoids using the simplex algorithm.

[Hint (to be read backwards): stnemetats eht ylppa neht dna ,eulav lamitpo sti ot setaler margorp eht fo ytilibisaef eht taht os noitcnuf evitcejbo eht gnivlovni melborp lamirp eht ot tniartsnoc a ddA]