Algorithmic Combinatorics Wi 2018–19
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 13

Due date: 16:15, 12th February

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 A father plays Rock-Paper-Scissors with his daughter. However, there is a slight twist – while he can choose any of rock, paper or scissors, she is only allowed to choose rock or paper. The rules otherwise remain the same: paper beats rock, rock beats scissors, and scissors beats paper. Determine the optimal strategies for each player in these conditions.

Exercise 2 Ada and Buu play a game. They each pick an integer in $\{1, 2, 3\}$. If the sum of their numbers is odd, Ada pays Buu €10, while if the sum is even, Buu gives Ada €10.

(a) Identify all the (mixed) Nash equilibria. What is the value of the game?

(b) Ada plays an optimal strategy, but notices that Buu is choosing his number uniformly at random. How should she adjust her strategy to take advantage of his mistake?

(c) After Ada makes this change, Buu starts losing money quickly. Observing that there are only 4 ways he can win, but 5 ways for Ada to win, he suggests that he should only have to pay €8 when Ada wins. Should Ada agree to these new terms?

Exercise 3 Let D be a finite collection of congruent closed disks in the plane, such that any two have a point in common. Show that D has a transversal of size at most 4.

[Hint (to be read backwards): ?noiger siht revoc ew nac woH ?detacol eb sertnec rehto eht nac erehW .)?yhw(ylevitcepser $(1, 0)$ dna $(-1, 0)$ ta dertnec era owt taht dna ,1 suidar evah sksid eht emussa nac eW]

Exercise 4 Recall the DLSZ lemma from the lecture, where we bounded the number of zeros a polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ inside a finite grid in terms of the *total* degree of f. In this exercise we will look at a variant involving individual degrees. Let $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a non-zero polynomial, and let d_i be the degree of f in the variable x_i; that is, d_i is the maximum power of x_i appearing in f. Let $S \subseteq \mathbb{F}$ be a finite set with $|S| \geq \max_i d_i$.

(a) Prove that f can have at most $|S|^n - \prod_{i=1}^n (|S| - d_i)$ zeroes in S^n.

(b) Give a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ (you can pick your favourite n), and a finite subset S of $\mathbb{R}$, such that the bound in (a) is sharp, whereas the bound from the lecture is not.

(c) Give a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ (you can again pick your n), and a finite subset S of $\mathbb{R}$, such that the bound from the lecture is sharp, whereas the bound in (a) is not.

Exercise 5 Let $k \geq 1$ be some integer, and let A and B be two $2^k \times 2^k$ matrices. We wish to efficiently compute $C = AB$. We express these in terms of $2^{k-1} \times 2^{k-1}$ submatrices:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}, \text{ and } C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}.$$

We now define some new matrices:

$$\begin{aligned} &M_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2}), && M_2 = (A_{2,1} + A_{2,2})B_{1,1}, && M_3 = A_{1,1}(B_{1,2} - B_{2,2}), \\ &M_4 = A_{2,2}(B_{2,1} - B_{1,1}), && M_5 = (A_{1,1} + A_{1,2})B_{2,2}, && M_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2}), \\ & && \text{and} && M_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2}). \end{aligned}$$

(a) Verify the identities $C_{1,1} = M_1 + M_4 - M_5 + M_7$, $C_{1,2} = M_3 + M_5$, $C_{2,1} = M_2 + M_4$, and $C_{2,2} = M_1 - M_2 + M_3 + M_6$.

(b) One can reuse these identities to calculate the products in the definition of the matrices M_i, leading to a recursive algorithm for computing the product $C = AB$. Estimate the running time (in terms of the number of arithmetic operations) of this algorithm.

(c) For general integers $n \geq 1$, how can this algorithm be applied to $n \times n$ matrices?

Exercise 6 In lecture we saw a randomised algorithm for determining if there is a perfect matching in a bipartite graph with n vertices in each part.

(a) Using this algorithm, explain how one can find a perfect matching, if it exists, in such a graph.

(b) If it takes $O(n^\omega)$ operations to find the determinant of an $n \times n$ matrix, how many operations does your matching-finding algorithm require?

Exercise 7 (Bonus 10 pnts) This exercise will show you how to extend our randomised perfect matching algorithm for bipartite graphs to general graphs. Let $G = ([n], E)$ be a graph on n vertices. As before, we introduce a new variable x_{ij} for each edge $\{i, j\} \in E$. The *Tutte matrix* A of the graph G is defined as $A = (a_{ij})_{i,j \in [n]}$, where

$$a_{ij} = \begin{cases} +x_{ij} & \text{if } i < j \text{ and } \{i, j\} \in E, \\ -x_{ji} & \text{if } i > j \text{ and } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find both the Tutte matrix A and its determinant $\det(A)$ when $G = K_3$ and $G = C_4$.

(b) Show $\det(A)$ is not the zero polynomial if G has a perfect matching.

We can think of a permutation $\pi \in S_n$ in terms of its cycle structure[1], which allows us to define $\operatorname{sgn}(\pi) = (-1)^{\# \text{ even cycles in } \pi}$. We then have $\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i \in [n]} a_{i\pi(i)}$.

(c) If we think of isolated edges as cycles of length two, show that any nonzero monomial in this expansion of $\det(A)$ corresponds to a partition of $[n]$ into vertex-disjoint cycles in G.

(d) By reversing the direction of an odd cycle, show that if $\det(A)$ is not the zero polynomial, then there is some partition of $[n]$ into vertex-disjoint cycles in G, all of which have even length.

(e) Deduce that $\det(A)$ is not the zero polynomial if and only if G has a perfect matching, and give a randomised algorithm for testing for the existence of a perfect matching in G.

[1] For example, if $n = 6$ and $\pi(1) = 2$, $\pi(2) = 5$, $\pi(3) = 4$, $\pi(4) = 3$, $\pi(5) = 1$ and $\pi(6) = 6$, then π has cycle structure (1 2 5) (3 4) (6).