Algorithmic Combinatorics | Wi 2018–19

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Exercise Sheet 14

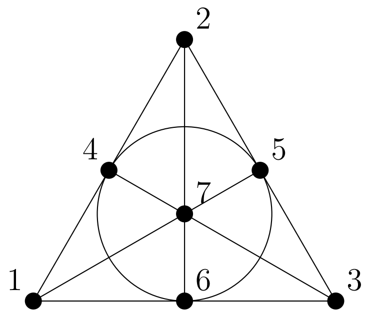
Due date: 16:15, 26th March

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 Prove that if H is k-uniform hypergraph with $|E(H)| \leq 2^{k-1}$, then H is 2-colorable.

Exercise 2

(1) Prove that the following hypergraph with vertices $\{1, 2, 3, 4, 5, 6, 7\}$ and hyperedges $\{124, 136, 157, 235, 267, 347, 456\}$ is non-two-colorable.



(2) Prove that every 3-uniform hypergraph with at most 6 edges is two-colorable.

Exercise 3 Show that the Bridge-It game by Gale (discussed in the lectures) cannot end in a draw.

Exercise 4 Let $\mathcal{F}$ be a k-uniform hypergraph on vertex set X with $|\mathcal{F}| > 2^{k-3} \cdot \Delta_2(\mathcal{F}) \cdot |X|$, where

$$\Delta_2(\mathcal{F}) = \max\{|A \in \mathcal{F} : x, y \in A| : x, y \in X, x \neq y\}.$$

Prove that $\mathcal{F}$ is Maker's win.

Exercise 5 In the strong (n,t)-Ramsey game, we have $X = E(K_n)$, and the winning sets are given by $\mathcal{F} = \{E(K) : K \text{ is a } t\text{-clique in } K_n\}$. We can think of this game as being played on the complete graph K_n; the players alternatively colour the edges of the complete graph, and the first player to create a monochromatic K_t wins the game.

(a) Prove that for every $t \geq 2$, there is some finite $n_0(t)$ such that whenever $n \geq n_0(t)$, First Player has a winning strategy for the strong (n,t)-Ramsey game.

(b) Show that $n_0(3) = 5$.

Exercise 6 Suppose we modify the rules of the Maker–Breaker game on a k-uniform hypergraph $(X, \mathcal{F})$: while Maker still selects one element of X in every turn, Breaker can now select q elements in every turn. Their goals remain the same: Maker wins if she selects all k elements from some winning set $F \in \mathcal{F}$, while Breaker wins if he selects at least one element from every winning set $F \in \mathcal{F}$.

Show that whenever $|\mathcal{F}| < (q+1)^{k-1}$, Breaker has a winning strategy, and describe this winning strategy explicitly.

[Hint (to be read backwards): .meroehT egdirfleS-sodrE eht fo foorp eht cimim ot yrt dna ,noitcnuf regnad nwo ruoy enifeD]