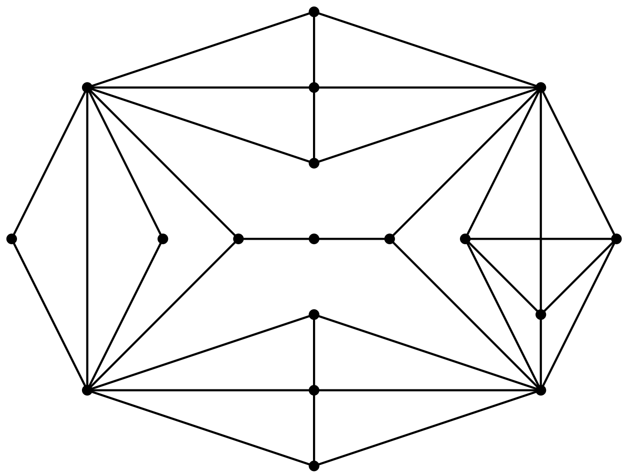
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 3

Due date: 16:15, 6th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 Determine the matching number of the following graph.



Exercise 2

(a) Prove that for any graph $G = (V, E)$, for every subset S of V, the parity of $o(G \setminus S) - |S|$ is equal to the parity of $|V|$, where $o(G \setminus S)$ is the number of odd components in the induced subgraph $G[V \setminus S]$.

(b) Prove that for any graph $G = (V, E)$, we have

$$2\alpha'(G) = \min_{S \subseteq V} \left(|V| + |S| - o(G \setminus S) \right),$$

where $\alpha'(G)$ is the maximum size of a matching in G.

[Hint (to be read backwards): d fo eulav nesohc llew a rof ,G hparg eht ot K_d euqilc a gnidda yb G' hparg yralixua na etaerc ,snoitcerid eht fo eno roF.]

Exercise 3

(a) Prove that if a 3-regular graph has strictly less than three cut edges, then it has a perfect matching.

(b) Give an example of a 3-regular graph with no perfect matching.

(c) For every $k \geq 2$, construct a k-regular graph with no perfect matching.

Exercise 4 Let T be a tree such that for every vertex v in T, the number of odd components in $T - v$ is equal to 1. Prove that T has a unique perfect matching.

Exercise 5 Given a graph $G = (V, E)$, one could try to apply the greedy algorithm to find a maximum matching of G. Order the edges $E = \{e_1, e_2, \ldots, e_m\}$ in some (arbitrary) way, and start with $M_0 = \emptyset$. At time t, for every $1 \leq t \leq m$, if $M_{t-1} \cup \{e_t\}$ is a matching, set $M_t = M_{t-1} \cup \{e_{t-1}\}$, and otherwise set $M_t = M_{t-1}$. Return the final matching M_m.

Prove that this gives a $\frac{1}{2}$-approximation algorithm for the maximum matching problem, and give an example to show that the $\frac{1}{2}$ approximation ratio is tight for this algorithm.

Exercise 6 A *matroid* is a pair $(X, \mathcal{B})$ of a finite ground set X and a collection $\mathcal{B}$ of subsets of X called *bases* that satisfy the following axioms:

(A1) There is at least one basis, i.e. $\mathcal{B} \neq \emptyset$.

(A2) The basis exchange property: If $A, B \in \mathcal{B}$ with $A \neq B$, then for every $a \in A \setminus B$ there is some $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ is another basis.

A set A is called *independent* if it is a subset of some basis; that is, if there is $B \in \mathcal{B}$ with $A \subseteq B$. You are already familiar with some fundamental matroids: for example, the set of bases of a finite vector space, or the set of spanning trees in a connected graph [1]. Start by showing the following is true in this more general setting.

(a) Show that all bases in a matroid must have the same cardinality.

If we assign nonnegative weights $w : X \to \mathbb{R}_{\geq 0}$ to elements in the ground set, we can then ask for the minimum weight basis; that is, for $B \in \mathcal{B}$ minimising $\sum_{x \in B} w(x)$.

The greedy algorithm is as follows: let X be ordered by weight, so $X = \{x_1, x_2, \ldots, x_n\}$ with $w(x_1) \leq w(x_2) \leq \ldots \leq w(x_n)$. Start with $S_0 = \emptyset$. At time t, for $1 \leq t \leq n$, let $T = S_{t-1} \cup \{x_t\}$. If T is independent, set $S_t = T$, and otherwise set $S_t = S_{t-1}$. The output of the algorithm is the final set S_n.

(b) Prove that the greedy algorithm produces a basis of minimum weight.

(**Bonus 10 pts**) Prove that the converse of (b) is also true: if $\mathcal{F}$ is a collection of subsets of a finite ground set X such that for any nonnegative weight function $w : X \to \mathbb{R}_{\geq 0}$, the greedy algorithm always produces a set $F \in \mathcal{F}$ of minimal weight, then $(X, \mathcal{F})$ is a matroid.

[1] Try to prove this