Algorithmic Combinatorics Wi 2018–19

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Exercise Sheet 4

Due date: 16:15, 13th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 We are given a complete graph $K_{n,n}$, with vertex classes $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, together with nonnegative edge weights $\omega_{i,j} = \omega(\{x_i, y_j\}) \geq 0$ for all $1 \leq i, j \leq n$. We also have price functions for the vertices, with $u(x_i) = u_i$ and $v(y_j) = v_j$. Such a pricing is called a weighted cover if $u_i + u_j \geq \omega_{i,j}$ for all $1 \leq i, j \leq n$.

Prove that for every perfect matching M and every weighted cover (u, v), we have $\omega(M) \leq c(u, v)$, where $\omega(M) = \sum_{e \in M} \omega(e)$ and $c(u, v) = \sum_{i=1}^{n} (u_i + v_i)$. Moreover, prove that we have equality if and only if there is some permutation $\pi \in S_n$ such that $M = \{\{x_i, y_{\pi(i)}\} : i \in [n]\}$ and $u_i + v_{\pi(i)} = \omega_{i,\pi(i)}$ for all i.

Exercise 2 Use the Hungarian Algorithm, showing all the key steps, to find a *minimum*-weight matching in $K_{5,5}$ with the edge weights $W = (\omega_{i,j})$ as below, and then give a short proof that your matching is optimal.

$$W = \begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}.$$

Exercise 3 Prove that in the Hungarian algorithm the value of the cost function decreases at each iteration (except for the last one). Use this to give a different proof of the termination of the Hungarian algorithm, in the special case when all the weights on the edges are rational numbers.

Exercise 4 Prove that if a graph on $2n$ vertices has a unique perfect matching, then it has at most n^2 edges. Show that this bound is tight.

Exercise 5 Let k, n be positive integers with $n \geq k + 1$. Prove that if a graph G on n vertices has $\delta(G) \geq (n + k - 2)/2$, then G is k-connected.

(**Bonus**) Prove that this bound is best possible.