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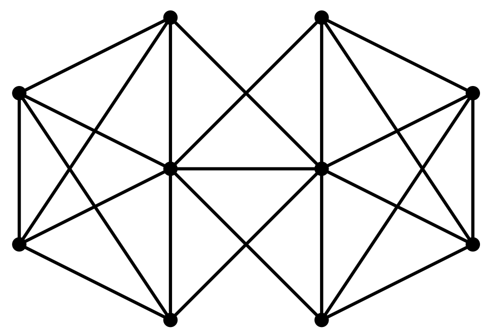
# Exercise Sheet 5

**Due date: 16:15, 20th November**

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

**Exercise 1** Determine $\kappa(G)$ and $\kappa'(G)$ for the following graph:



**Exercise 2** Let $n, k \in \mathbb{N}$, with $n \geq k+1 \geq 3$, and define the graph $H_{k,n}$ as follows. Place $n$ vertices around a circle, equally spaced. If $k$ is even, form $H_{k,n}$ by adding an edge between each vertex and its nearest $k/2$ vertices in each direction around the circle. If $k$ is odd and $n$ is even, form $H_{k,n}$ by adding an edge between each vertex and its nearest $(k-1)/2$ vertices in each direction around the circle, and an edge between every pair of diametrically opposite vertices. If both $k$ and $n$ are odd, then label the vertices by residue classes modulo $n$, and form $H_{k,n}$ from $H_{k-1,n}$ by adding the edges between $i$ and $i+(n-1)/2$ for $0 \leq i \leq (n-1)/2$.

Prove that $\kappa(H_{k,n}) = k$, and deduce that the minimum number of edges in a $k$-connected graph on $n$ vertices, for $k \geq 2$, is $\lceil (kn)/2 \rceil$.

**Exercise 3** For every set of integers $k, k', d$ satisfying $1 \leq k \leq k' \leq d$, construct a graph $G$ with $\kappa(G) = k$, $\kappa'(G) = k'$ and $\delta(G) = d$.

**Exercise 4** Prove that for every graph $G$ of maximum degree $\Delta(G) \leq 3$, $\kappa(G) = \kappa'(G)$. Deduce that the Petersen graph has edge connectivity number equal to 3.

**Exercise 5** Recall that for vertices $x, y \in V(G)$, $\kappa'(x, y)$ is the minimum number of edges that have to be deleted from $G$ to separate $x$ from $y$, and $\lambda'(x, y)$ is the maximum size of a set of pairwise edge-disjoint $x$–$y$ paths.

(a) Using the Local Vertex Menger Theorem, prove that for every pair of vertices $x, y \in V(G)$, we have $\kappa'(x, y) = \lambda'(x, y)$.

(b) Deduce that $G$ is $k$-edge-connected if and only if for every pair $x, y$ of vertices, there is a set of $k$ pairwise-edge-disjoint $x$–$y$ paths.

**Exercise 6** For any directed graph $\vec{D}$ and vertex set $\emptyset \neq S \subsetneq V(\vec{D})$, the edge-cut $[S, \bar{S}]$ is the set of edges starting in $S$ and ending in $\bar{S} = V(\vec{D}) \setminus S$. Prove that for any network $(\vec{D}, s, t, c)$, feasible flow $f : \vec{E}(\vec{D}) \to \mathbb{R}_{\geq 0}$ and vertex set $S$ such that $s \in S$ and $t \notin S$, the value of the flow is given by

$$\mathrm{val}(f) = \sum_{\vec{e} \in [S, \bar{S}]} f(\vec{e}) - \sum_{\vec{e} \in [\bar{S}, S]} f(\vec{e}).$$