Algorithmic Combinatorics Wi 2018–19
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Exercise Sheet 5

Due date: 16:15, 27th November

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 Use the Ford-Fulkerson theorem on network flows to prove that the matching number of a bipartite graph is equal to its vertex cover number.

Exercise 2 Let G be a directed graph, and let x, y be two vertices such that $(x, y) \notin E(G)$. A set $S \subseteq V(G) \setminus \{x, y\}$ is called an x, y-cut if $G - S$ has no directed path from x to y. Define $\kappa_D(x, y) = \min\{|S| : S \text{ is an } x, y\text{-cut}\}$ and $\lambda_D(x, y) = \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. paths from } x \text{ to } y\}$. Using the Ford-Fulkerson theorem, prove that

$$\kappa_D(x, y) = \lambda_D(x, y).$$

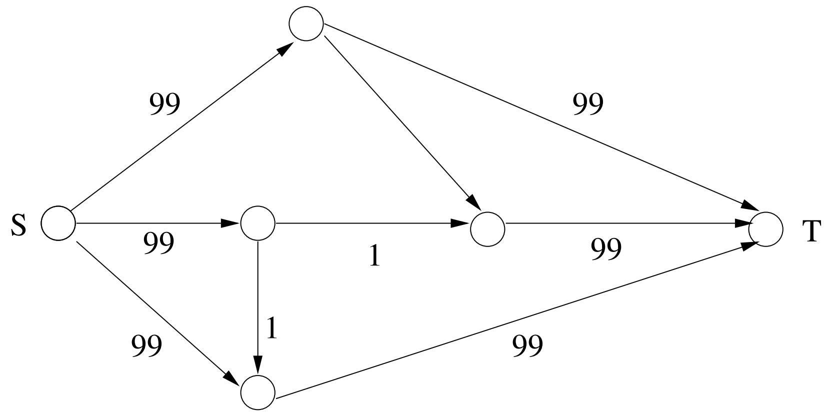
Exercise 3 The following table describes a network. Find a flow of maximum possible value from "Dub" to "Lit". [1]

Start	→	End	Capacity (€, mil)	Start	→	End	Capacity (€, mil)
Dub	→	Ams	50	Dub	→	Mal	140
Ams	→	Jer	40	Ams	→	Lux	40
Ams	→	Ber	50	Jer	→	Lux	30
Mal	→	Ams	130	Mal	→	Zür	20
Mal	→	Cay	50	Ber	→	Mal	60
Ber	→	Zür	60	Lux	→	Ber	40
Lux	→	Zür	30	Lux	→	Lit	60
Cay	→	Zür	30	Zür	→	Lit	120

Exercise 4 Let G be an edge-weighted graph. Let the *value* of a spanning tree of G be the minimum weight appearing on its edges. Let the *cap* from an edge cut $[S, \overline{S}]$ be the maximum weight appearing on its edges. Prove that the maximum value of a spanning tree is equal to the minimum cap of an edge cut in G.

[1] Non mathematical **Bonus**, figure out the story behind this network

Exercise 5 Consider the network in the figure above. The source and the sink are marked with S and T, and the capacities of all but one edge are indicated. The remaining edge has capacity $r = \frac{1}{2}(\sqrt{5}-1)$.



(a) Find (with proof) the value of the maximum flow in the network.

(b) Describe a choice of augmenting paths in the Ford-Fulkerson algorithm for which the algorithm never finishes and the flow value converges to $2+\sqrt{5}$.

[Hint (to be read backwards): !redro thgir eht ni meht ylppa ot erus eB .evif htgnel fo eno dna ruof htgnel fo owt ,shtap gnitnemgua tcnitsid eerht ylppa yldetaeper dluohs uoY]

[Hint (to be read backwards): .$a_n = r^n$ yb deifsitas noitaler ecnerrucer eht dniF]

Exercise 6 **(Bonus)** Suppose $n \geq 2$. Baranyai's Theorem guarantees $\binom{[3n]}{3}$ can be partitioned into perfect matchings without explicitly describing these matchings. In this exercise you will give such an explicit description in the case when $p = 3n-1$ is a prime number.

(a) Consider the field $\mathbb{F}_p$, and denote by $\mathbb{F}_p^*$ the set of invertible elements, namely $\mathbb{F}_p^* = \{1, 2, \ldots, p-1\}$. Define the map $\pi : \mathbb{F}_p^* \to \mathbb{F}_p$ by $\pi(x) = -(1+x)x^{-1}$. Show that π is injective and $\pi^3(x) = x$ for any $x \neq p-1$.

(b) Add a new element u to $\mathbb{F}_p$, and extend π to $\{u, 0\}$ injectively so that $\pi^3(x) = x$ for all $x \in \mathbb{F}_p \cup \{u\}$. Show that this gives some perfect matching M_0 in $\binom{[3n]}{3}$.

(c) By considering affine transformations $x \mapsto ax+b$, find another $\binom{3n-1}{2} - 1$ perfect matchings in $\binom{[3n]}{3}$.

(d) Show that these matchings partition $\binom{[3n]}{3}$ into perfect matchings.