Algorithmic Combinatorics Wi 2018–19
Tibor Szabó
Anurag Bishnoi

Exercise Sheet 7

Due date: 12:30, 5th December

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1 In this exercise you will finish the proof of Baranyai's theorem using network flows that was discussed in the lectures.

Recall that for $1 \leq \ell \leq n$, we sought a collection of $M = \binom{n-1}{k-1}$ m-partitions $\mathcal{A}_i$ of $[\ell]$[1], where $m = \frac{n}{k}$, such that every set $F \subseteq [\ell]$ appeared (with multiplicity) in exactly $\binom{n-\ell}{k-|F|}$ of the m-partitions. We obtain such a partition via induction on ℓ, where the base case of $\ell = 0$ is true by taking each $\mathcal{A}_i$ to be m copies of $\emptyset$. Now, for the induction step, given such a collection of m-partitions for $\ell \leq n-1$, we built a network $(\vec{D}, s, t, c)$, with $\vec{D}$ as a directed multigraph, where $V(\vec{D}) = \{s,t\} \cup \{\mathcal{A}_i : i \in [M]\} \cup \{F : F \subseteq [\ell]\}$ and

$$\vec{E}(\vec{D}) = \{(s, \mathcal{A}_i) : i \in [M]\} \cup \{(\mathcal{A}_i, F) : i \in [M], F \in \mathcal{A}_i\} \cup \{(F,t) : F \subseteq [\ell]\},$$

where between each $\mathcal{A}_i$ and $\emptyset$ we add as many multiple edges as the number of times $\emptyset$ appears in $\mathcal{A}_i$. The capacities were given by

$$c(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F,t) \\ M+1 & \text{otherwise} \end{cases} .$$

(a) Prove that the flow f defined by

$$f(\vec{e}) = \begin{cases} 1 & \vec{e} = (s, \mathcal{A}_i) \\ \frac{k-|F|}{n-\ell} & \vec{e} = (\mathcal{A}_i, F) \\ \binom{n-(\ell+1)}{k-(|F|+1)} & \vec{e} = (F,t) \end{cases}$$

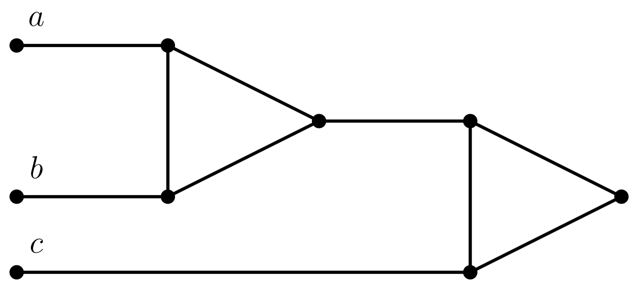
is a feasible maximum flow of value M.

(b) Use the integrality theorem to find a unique set $F_i \in \mathcal{A}_i$ for each $i \in [M]$. Then form the m-partitions $\mathcal{A}'_1, \ldots, \mathcal{A}'_M$ of $[\ell+1]$ by adding the element $\ell+1$ to the set F_i in each $\mathcal{A}_i$. Show that this collection of m-partitions of $[\ell+1]$ satisfies the required conditions.

[1] An m-partition of a set X is a multiset of pairwise disjoint subsets of X, some of which might be empty, whose union is X. Note that because of pairwise disjointness, only $\emptyset$ can occur with multiplicity more than 1 in an m-partition.

Exercise 2 Recall that a 3-CNF formula consists of clauses with exactly 3 literals, each corresponding to a different variable, where a literal is either a variable x_i or its negation $\neg x_i$. For example, $f(x_1, x_2, x_3, x_4, x_5) = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_3 \vee x_5) \wedge (x_1 \vee \neg x_4 \vee \neg x_5)$ is a 3-CNF. We say that a 3-CNF formula is satisfiable if there exists a True/False assignment to the variables that makes the formula True.

For a clause $C = a \vee b \vee c$, define the *OR-gadget* as the following graph, where the left-most vertices are the *input nodes* and the right-most vertex is called the *output node*.



Now let ϕ be a 3-CNF with $C_1, \ldots, C_m$ as its clauses and $x_1, \ldots, x_n$ as its variables. Construct a graph G as follows. First create a triangle in G with vertices labelled T, F, B. For every literal x_i, introduce two new vertices v_i and $\overline{v}_i$, and create a triangle $B, v_i, \overline{v}_i$. Now for each C_i create an OR-gadget, where the input nodes are the vertices corresponding to the literals appearing in C_i, and add an edge between the output node and F, and an edge between the output node and B.

(a) Prove that ϕ is satisfiable if and only if the graph G is 3-colorable.

(b) The 3-SAT problem is to decide whether a 3-CNF is satisfiable or not. We know from the Cook-Levin theorem that 3-SAT is NP-complete. Deduce that 3-colorability is NP-complete.

Exercise 3 Let G be a graph with the property that every two odd cycles in G always share a vertex. Prove that $\chi(G) \leq 5$.

(**Bonus**) Clearly K_5 shows that this bound is tight. Find a graph G with $\chi(G) = 4$ satisfying this property. Can you find a K_5-free graph with chromatic number 5 in which every two odd cycles share a vertex?

Exercise 4 Prove that any graph with chromatic number equal to k has at least $\binom{k}{2}$ edges. Deduce that a graph on m edges has chromatic number at most $O(\sqrt{m})$.

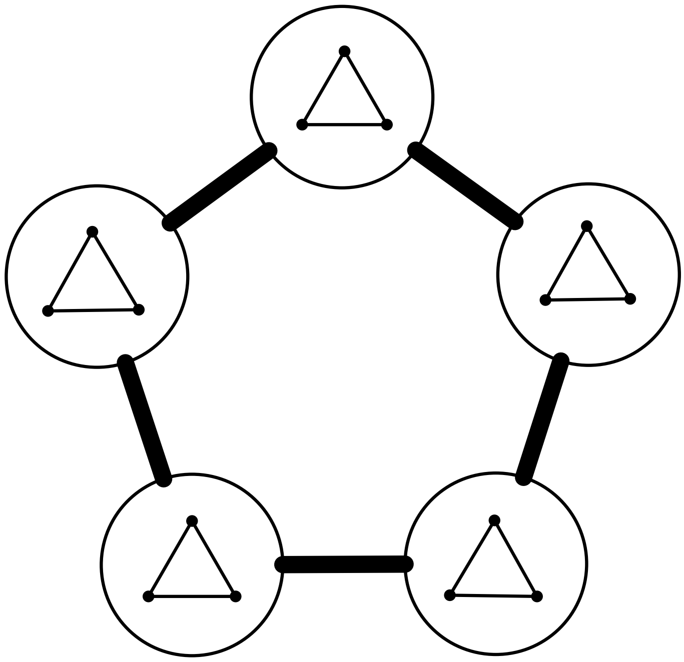
Exercise 5 Prove that for every triangle free graph G, the trivial upper bound $\chi(G) \leq \Delta(G) + 1$ can be strengthened to:

$$\chi(G) \leq 3 \left\lceil \frac{\Delta(G) + 1}{4} \right\rceil .$$

[Hint (to be read backwards): 3 tsom ta rebmun citamorhc sah strap eht fo hcae no hpargbus decudni eht taht wohs dna desiminim si trap hcae nihtiw segde fo rebmun eht taht hcus strap fo rebmun $\lceil(\Delta(G)+1)/4\rceil$ otni secitrev fo tes eht noititraP]

Exercise 6 In this exercise we will show that Hajós' conjecture is false in general. Recall that the conjecture states that for any graph G and any positive integer k, $\chi(G) \geq k$ implies that G contains a K_k subdivision.

(a) Show that for the following 15 vertex graph G, we have $\chi(G) = 8$ and G contains no subdivisions of K_8. Here thick edges between the circles mean that every vertex in one circle is adjacent to every vertex in the other.



(b) Prove that Hajós' conjecture is false for all $k \geq 8$.

(c) (**Bonus**) Construct a graph with chromatic number 7 that does not contain any K_7 subdivision.