Algorithmic Combinatorics Wi 2018–19
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Exercise Sheet 9

Due date: 16:15, 18th December

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution. Until submission, you are forbidden to look at the solutions of any of these exercises on the internet.

Exercise 1

(a) Give an explicit edge colouring of $K_{m,n}$ to show that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.

(b) Let G be a simple bipartite graph. Construct a simple bipartite $\Delta(G)$-regular graph H that contains G as a subgraph. (In the lecture we proved the analogous statement for multigraphs.)

Exercise 2 Adapt the proof of Vizing's theorem to obtain a polynomial time algorithm to properly edge colour a graph G with $\Delta(G) + 1$ colours.

(Bonus) Reduce the worst case time complexity of your algorithm as much as you can.

Exercise 3 Let G be a simple graph.

(a) Prove that the number of edges in $L(G)$ is equal to $\sum_{v \in V(G)} \binom{\deg(v)}{2}$.

(b) Prove that $G \cong L(G)$ if and only if G is 2-regular.

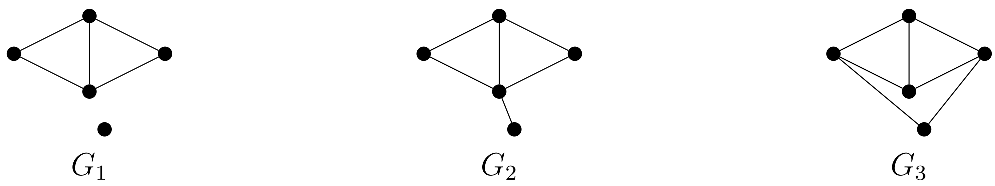
Exercise 4 A *clique decomposition* of a multigraph G is a collection $\mathcal{K}$ of cliques[1] in G such that every edge of G is contained in at least one clique in $\mathcal{K}$.

(a) Prove that G is the line graph of some multigraph H if and only if G has a clique decomposition $\mathcal{K}$ with every vertex appearing in exactly two cliques in $\mathcal{K}$. Show further that H may be taken to be a simple graph if and only if every pair of vertices in G appear together in at most one clique of $\mathcal{K}$.[2]

[1] A clique is a complete subgraph, and we allow cliques with just a single vertex.

[2] Note that this characterisation is not that useful (at least immediately) for designing an efficient algorithm to check whether a graph is the line graph of a simple graph or not, but there are other characterisations known that yield efficient algorithms.

(b) For the three graphs G_i below, provide, if possible, a multigraph H_i for which we have $G_i = L(H_i)$. If possible, make H_i a simple graph. Whenever these tasks are impossible, explain why.



Exercise 5 Recall that Brook's theorem states that $\chi(G) \leq \Delta(G)$ for every graph G which is not a clique or an odd cycle. Use Brook's theorem to prove Vizing's theorem for graphs with maximum degree 3.

Exercise 6 We saw that Petersen graph is a 3-regular graph with edge chromatic number equal to 4. In this exercise we will construct infinitely many 3-regular graphs whose edge chromatic number is equal to 4.

Let $n \geq 3$ be an odd integer, and construct the graph G_n as follows. Take n pairwise disjoint copies of $K_{1,3}$, i.e., the star graph with 4 vertices, and denote the four vertices of the i-th copy by a_i, b_i, c_i and d_i where a_i is the degree 3 vertex (the center of the star). Add n edges to make $b_1, \ldots, b_n$ into a cycle and $2n$ edges to make $c_1, \ldots, c_n, d_1, \ldots, d_n$ into a cycle (in that order). For every such n, prove the following.

(a) $\chi'(G_n) = 4$.

(b) $\kappa'(G_n) > 1$.

(c) (**Bonus**) G_n is non-planar.[3]

[3] A class two 3-regular graph with $\kappa' > 1$ is known as a *Snark*. Until 1970's, there were only 5 Snark's known, Petersen graph being one of them, and then several infinite families (like the one above) were constructed. It has been known since 1880 that the four colour theorem is equivalent to the statement that every snark is non-planar. But alas, this did not help in proving the four colour theorem.