Connectivity

A separating set (or vertex cut) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ is disconnected. For $G \neq K_n$, the connectivity of G is

$$\kappa(G) := \min\{|S| : S \text{ is a vertex cut}\}.$$

By definition, $\kappa(K_n) := n - 1$.

A graph G is called k-connected if $v(G) \geq k + 1$ and there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Initial bounds: $\kappa(G) \leq v(G) - 1$ (equality only for K_n)
$\kappa(G) \leq \delta(G)$

Examples: $\kappa(K_{n,m}) = \min\{n, m\}$
$\kappa(Q_d) = d$

Extremal problem: What is the minimum number of edges in a k-connected graph?

Theorem. For every n, the minimum number of edges in a k-connected graph is $\lceil kn/2 \rceil$.
Proof: HW

A sufficient condition for Hamiltonicity via κ—

Theorem. (Erdős-Chvátal, 1972) If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. (Unless $G = K_2$)

Proof. Let $k = \kappa(G) > 1$. Let $C = (v_1, \ldots v_\ell)$ be the longest cycle.

$\delta(G) \geq k \Rightarrow length(C) \geq k + 1$

Let H be a component of $G - C$.

Let $v_{i_1}, \ldots v_{i_k} \in V(C)$ be vertices with an edge to $V(H)$. Then:

- $U = \{v_{i_1+1}, \ldots, v_{i_k+1}\}$ is independent
- No edge between U and $V(H)$.

$\Rightarrow \alpha(G) \geq k + 1$. $\square$

Edge-connectivity

Def. A set $F \subseteq E(G)$ of edges of a multigraph G is a disconnecting set if $G - F$ is disconnected. The edge-connectivity of G is

$$\kappa'(G) := \min\{\ |F|\ : F \text{ is a disconnecting set}\}.$$

A graph G is called k-edge-connected if $\kappa'(G) \geq k$.

An edge cut of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

..., where for $S, T \subseteq V(G)$, $[S,T] := \{xy \in E(G) : x \in S, y \in T\}$.

Obs. A minimal disconnecting set is an edge cut.
In particular,

$$\kappa'(G) = \min\{\ |[S, \bar{S}]|\ : \emptyset \subset S \subset V(G)\}.$$

and G is k-edge-connected iff there is no edge cut of size $\leq k - 1$.

Theorem. (Whitney, 1932) If G is a simple graph, then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Recall: Characterization of 2-connectivity

Decision problem: “Is G k-connected?” is in co-NP.
Is the problem also in NP? How about P?

Remark. k-connectivity is in P when k is a constant: One checks for each subset of size $\leq (k-1)$ whether its deletion results in a disconnected graph. (There are polynomially many subsets to check, each check is done by BFS or DFS in poly-time.)
But this does **not** work when $k = k(n)$ is a function of n tending to ∞. (The number of subsets to check is superpolynomial.)

An NP-co-NP-characterization of k-connectivity?

For $k = 2$: a simple sufficient condition, which prevents that the removal of a single vertex disconnects a graph G, is that for any pair $u, v \in V(G)$ there are **two disjoint ways** to get from u to v.

Surprisingly, this condition is also necessary!
Theorem. (Whitney,1932) A graph G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G.

Proof: Create two internally disjoint u, v-paths using induction on $dist(u, v)$ (the length of a shortest u, v-path). $\square$

Corollary 2-connectivity is in NP$\cap$co-NP.

A strengthening of Whitney's Thm.
A graph G is 2-connected iff $\delta(G) \geq 1$ and every pair of edges of G lies on a common cycle.

> **Expansion Lemma.** Let G' be a supergraph of a k-connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors.
> Then G' is k-connected as well.

An obvious way to generalize Whitney's sufficient condition in order to ensure k-connectivity is if we require that between any two vertices there are k disjoint ways to get from one to the other. This also turns out to be necessary, but the proof is much less obvious!

(Global-Vertex)-Menger Theorem. A graph G is k-connected iff for every $u, v \in V(G)$ there exist k pairwise internally disjoint u, v-paths in G.

Corollary "k-connectivity" is in NP$\cap$co-NP for any function $k = k(n)$

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separating set if $G - S$ has no x, y-path.

A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.

Define

$\kappa(x, y)$:= min$\{|S|$: S is an x, y-separating set,$\}$ and
$\lambda(x, y)$:= max$\{|\mathcal{P}|$: $\mathcal{P}$ is a set of p.i.d. x, y-paths$\}$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x, y) = \lambda(x, y).$$

Proof. Coming soon. (Using much more general machinery.)

Corollary (Global Vertex-Menger Theorem) A graph G is k-connected iff for any two vertices $x, y \in V(G)$ there exist a set of k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if $G - F$ has no x, y-path. Define

$$\kappa'(x,y) := \min\{|F| : F \text{ is an } x,y\text{-disconnecting set,}\}$$
$$\lambda'(x,y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* \ x,y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

Local Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x,y) = \lambda'(x,y).$$

Proof. HW

Corollary (Global Edge-Menger Theorem) Multigraph G is k-edge-connected iff there is a set of k p.e.d.x, y-paths for any two vertices x and y.

Corollary "k-edge-connectivity" is in NP∩co-NP for any function $k = k(n)$

Network flows

Network (D, s, t, c), where
$D = (V, E)$ is a directed multigraph,
$s \in V$ is the source, $t \in V$ is the sink,
$c : E \to \mathbb{R}_{\geq 0}$ is the capacity function.

A function $f : E \to \mathbb{R}$ is called a flow. Define

$$f^+(v) := \sum_{e^- = v} f(e)$$

$$f^-(v) := \sum_{e^+ = v} f(e), \text{ where } e = (e^-, e^+).$$

Flow f is feasible if

(i) $f^+(v) = f^-(v)$ for every $v \neq s, t$ (conservation constraints), and

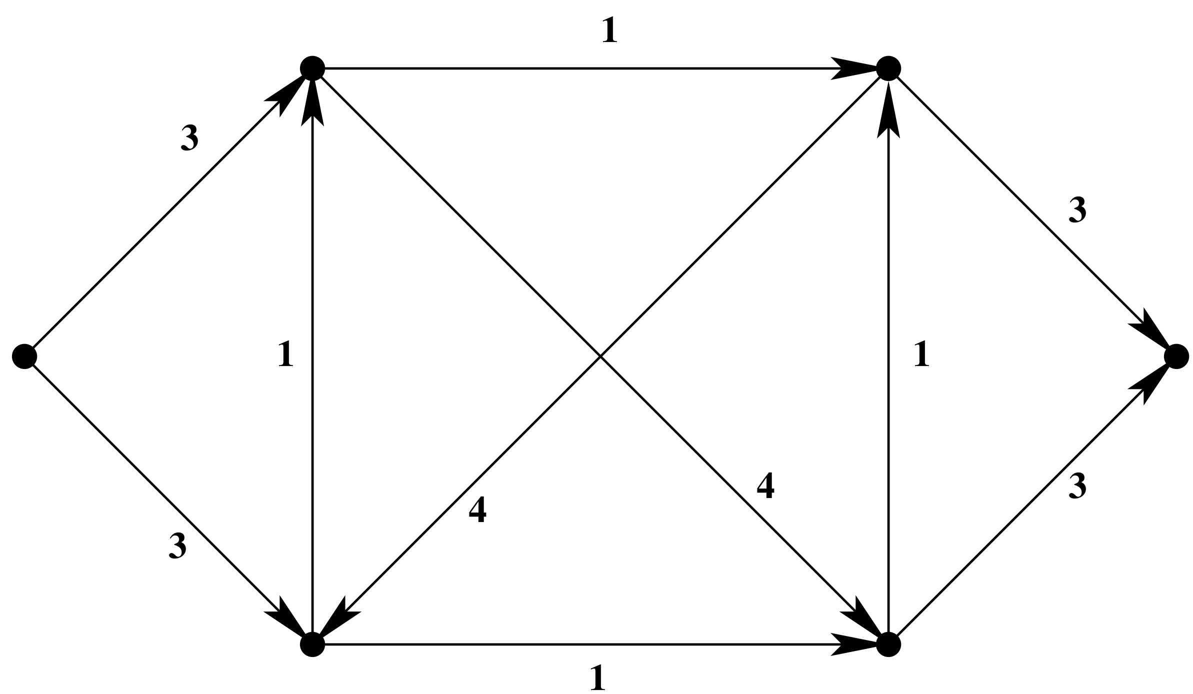
(ii) $0 \leq f(e) \leq c(e)$ for every $e \in E$ (capacity constraints).

value of flow, $val(f) := f^-(t) - f^+(t)$.

WANT:
A maximum flow: feasible flow with maximum value

Example: finding a max flow

Starting with the 0-flow



A way to **prove** maximality of a flow $\longrightarrow$

$\longrightarrow$ capacity of source/sink cuts

Source/sink cuts

$[S, \bar{S}] := \{(u, v) \in E(D) : u \in S, v \in \bar{S}\}$ is a source/sink cut if $s \in S$ and $t \in \bar{S}$

capacity of cut: $cap(S, \bar{S}) := \sum_{e \in [S,\bar{S}]} c(e)$.

Weak Duality Lemma. If f is a feasible flow and $[S, \bar{S}]$ is a source/sink cut, then

$$val(f) \leq cap(S, \bar{S}).$$

Proof.
$$\begin{aligned} cap(S, \bar{S}) &= \sum_{e \in [S,\bar{S}]} c(e) \\ &\geq \sum_{e \in [S,\bar{S}]} f(e) \\ &\geq \sum_{e \in [S,\bar{S}]} f(e) - \sum_{e \in [\bar{S},S]} f(e) \\ &= val(f). \square \end{aligned}$$

We used the capacity constraints and the feeling that the last equality must be true by the conservation constraints ... Proof (HW?)

The value of a feasible flow

Conservation Lemma. If f is any feasible flow, $s \in Q$, $t \notin Q$, then

$$\sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) = val(f).$$

Proof. By induction on $|\bar{Q}|$. If $|\bar{Q}| = 1$ then $\bar{Q} = \{t\}$ and by definition $f^-(t) - f^+(t) = val(f)$.

Let $|\bar{Q}| \geq 2$ and let $x \in \bar{Q}$, $x \neq t$.
Define $R = Q \cup \{x\}$. Since $|\bar{R}| < |\bar{Q}|$, by induction

$$\begin{aligned}
val(f) &= \sum_{e \in [R,\bar{R}]} f(e) - \sum_{e \in [\bar{R},R]} f(e) \\
&= \sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) + \sum_{u \in Q} f(xu) \\
&\quad - \sum_{u \in Q} f(ux) + \sum_{v \in \bar{R}} f(xv) - \sum_{v \in \bar{R}} f(vx) \\
&= \sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) + f^+(x) - f^-(x)
\end{aligned}$$

Remark. $val(f) = f^+(s) - f^-(s)$.

Improving a feasible flow: f-augmenting paths

G: underlying undirected multigraph of network D

s, t-path $s = v_0, e_1, v_1, e_2 \ldots v_{k-1}, e_k, v_k = t$ in G is an f-augmenting path, if for every i

- $f(e_i) < c(e_i)$ if e_i is a "forward edge"
- $f(e_i) > 0$ if e_i is a "backward edge"

Tolerance of the path P is $\min\{\epsilon(e) : e \in E(P)\}$, where $\epsilon(e) = c(e) - f(e)$ if e is forward, and
$\epsilon(e) = f(e)$ if e is backward.

Augmenting Lemma. Let f be feasible and P be an f-augmenting path with tolerance z. Define
$f'(e) := f(e) + z$ if e is forward,
$f'(e) := f(e) - z$ if e is backward.
$f'(e) := f(e)$ if $e \notin E(P)$,
Then f' is feasible with $val(f') = val(f) + z$.

Max Flow-Min Cut Theorem

Cut Lemma. For a feasible flow f define the subset

$S_f := \{v \in V : \exists\ f$-augmenting path* from s to $v\}$.

If $t \notin S$, then

$$cap(S_f, \bar{S}_f) = \sum_{e \in [S_f, \bar{S}_f]} f(e) - \sum_{e \in [\bar{S}_f, S_f]} f(e).$$

Max Flow-Min Cut Theorem (Ford-Fulkerson, 1956)

$$\max val(f) = \min cap(S, \bar{S}).$$

Proof.

$\leq$: Weak Duality.

$\geq$: Let g be a max flow. Then g has no augmenting path, so $t \notin S_g$, and then by the Cut Lemma and the Conservation Lemma

$$\begin{aligned} cap(S_g, \bar{S}_g) &= \sum_{e \in [S_g, \bar{S}_g]} g(e) - \sum_{e \in [\bar{S}_g, S_g]} g(e) \\ &= val(g). \square \end{aligned}$$

Application 1: Edge-Menger Theorem

Local-Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x, y) = \lambda'(x, y).$$

Proof. $\boxed{\leq}$ Build network (D, x, y, c) where
$V(D) := V(G)$
$E(D) := \{(u, v), (v, u) : uv \in E(G)\}$ and
$c(e) := 1$ for all $e \in E(D)$.

• For any $S \subset V$ with $x \in S$ and $y \notin S$, we have $|[S, \bar{S}]| = cap(S, \bar{S})$. Hence
$\kappa'(x, y) = \min cap(S, \bar{S}) = \max val(f)$.

• each set of p.e.d. path determines a feasible flow of value $\lambda'(x, y) \leq \max val(f)$.

A *unit flow* is a feasible flow that has value 1 along an s, t-path and 0 everywhere else.

Unit Flows Lemma. If f is a feasible flow with integer values, then there exists $m := val(f)$ unit flows $g_1, \ldots g_m$, such that $f = g_1 + \cdots + g_m$.

• We know that $\max val(f) = \kappa'(x, y)$ is an integer. But, is there a flow with integer values that realizes this??? (and hence is the sum of $\kappa'(x, y)$ unit flows?)

Characterization of maximum flows

Algorithm: Try to find a max flow with integer values by starting with the 0-flow and iteratively increasing its value, using augmenting paths, always by an integer.

• Tolerance of an augmenting path is an integer once the flow values and the capacities are integers.

• Maximum is indeed reached once there is no augmenting path.

Characterization Lemma. Feasible flow f is of maximum value iff there is NO f-augmenting path.

Proof. $\boxed{\Rightarrow}$ Augmenting Lemma.
$\boxed{\Leftarrow}$ If f has no augmenting path, then $t \notin S_f$ and by the Cut Lemma and the Conservation Lemma

$$\begin{aligned} cap(S_f, \bar{S}_f) &= \sum_{e \in [S_f, \bar{S}_f]} g(e) - \sum_{e \in [\bar{S}_f, S_f]} f(e) \\ &= val(f), \end{aligned}$$

so f is a max flow by Weak Duality. $\square$

Ford-Fulkerson Algorithm

Initialization $f \equiv 0$
WHILE there exists an augmenting path P
    DO augment flow f along P
return f

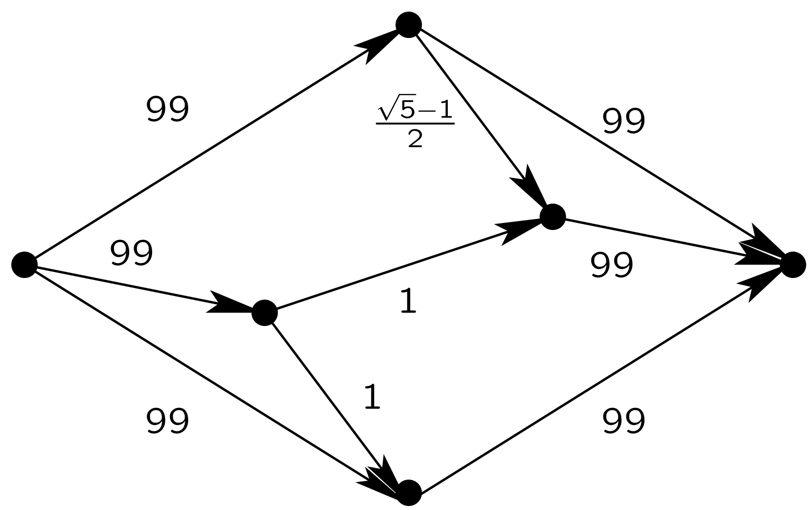
Corollary. (**Integrality Theorem**) If all capacities of a network are integers, then there is a maximum flow assigning integral flow to each edge.
Furthermore, some maximum flow can be partitioned into flows of unit value along path from source to sink.

Running times:

- Basic (careless) Ford-Fulkerson: might not even terminate, flow value might not converge to maximum;
  when capacities are integers, it terminates in time $O(m\,|f^*|)$, where f^* is a maximum flow.
- Edmonds-Karp: chooses a *shortest* augmenting path; runs in $O(nm^2)$

Example

The Max-flow Min-cut Theorem is true for real capacities as well,
BUT our algorithm might fail to find a maximum flow!!!



Example of Zwick (1995)

Remark. The max flow is 199. There is such an unfortunate choice of a sequence of augmenting paths, by which the flow value never grows above $2 + \sqrt{5}$.

Application 2: Menger's Theorem

Recall:

$\kappa(x,y)$:= min$\{|S|$: S is an x,y-cut,$\}$ and
$\lambda(x,y)$:= max$\{|\mathcal{P}|$: $\mathcal{P}$ is a set of p.i.d. x,y-paths$\}$

Local-Vertex-Menger Theorem Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x,y) = \lambda(x,y).$$

Proof. We apply the Integrality Theorem for the auxiliary network (D, x^+, y^-, c).

$$\begin{aligned} V(D) &:= \{v^-, v^+ : v \in V(G)\} \\ E(D) &:= \{(u^+v^-) : uv \in E(G)\} \\ &\quad \cup\{(v^-v^+) : v \in V(G)\} \end{aligned}$$

$c(u^+v^-) = \infty^*$ and $c(v^-v^+) = 1$.

*or rather a large enough **integer**, say $|V(D)|$.

Application 3: Baranyai's Theorem

$\chi'(K_n) = n-1$ is saying: $E(K_n)$ can be decomposed into pairwise disjoint perfect matchings.

k-uniform hypergraphs? $E(\mathcal{K}_n^{(k)}) = \binom{[n]}{k}$

Let $k|n$. $\mathcal{S} = \{S_1, \dots, S_{n/k}\}$ is a "perfect matching in $\mathcal{K}_n^{(k)}$ if $S_i \cap S_j = \emptyset$ for $i \neq j$.

There are perfect matchings in $\mathcal{K}_n^{(k)}$. (How many?)

Is there a decomposition of $\binom{[n]}{k}$ into perfect matchings?

Not obvious already for $k = 3$ (Peltesohn, 1936)

$k = 4$ (Bermond)

Theorem (Baranyai, 1973) For every $k|n$, there is a decomposition of $\binom{[n]}{k}$ into perfect matchings.

Proof of Baranyai's Theorem

Induction on the size of the underlying set $[n]$.
NOT the way you would think!!!

We imagine how the $m = \frac{n}{k}$ pairwise disjoint k-sets in each of the $M = \binom{n-1}{k-1} = \binom{n}{k}/m$ "perfect matchings" would develop as we add one by one the elements of $[n]$.

A **multi**set $\mathcal{A}$ is an m-partition of the base set X if $\mathcal{A}$ contains m pairwise disjoint sets whose union is X.

Remarks
An m-partition is a "perfect matching" in the making.
Pairwise disjoint $\Rightarrow$ only $\emptyset$ can occur more than once.

Stronger Statement For every l, $0 \leq l \leq n$ there exists M m-partitions of $[l]$, such that every set S occurs in $\binom{n-l}{k-|S|}$ m-partitions ($\emptyset$ is counted with multiplicity).

Remark For $l = n$ we obtain Baranyai's Theorem since $\binom{0}{k-|S|} = 0$ unless $|S| = k$, when its value is 1.

Proof of Stronger Statement: Induction on l.

$l = 0$: Let all $\mathcal{A}_i$ consists of m copies of $\emptyset$.

$l = 1$: Let all $\mathcal{A}_i$ consists of $m-1$ copies of $\emptyset$ and 1 copy of $\{1\}$.

Let $\mathcal{A}_1, \ldots, \mathcal{A}_M$ be a family of m-partitions of $[l]$ with the required property.
We construct one for $l+1$.

Define a network D:

$$V(D) = \{s, t\} \cup \{\mathcal{A}_i : i = 1, \ldots, M\} \cup 2^{[l]}.$$

$$E(D) = \{s\mathcal{A}_i : i \in [M]\} \cup \{\mathcal{A}_i S : S \in \mathcal{A}_i\}$$
$$\cup \{St : S \in 2^{[l]}\}.$$

Edge $\mathcal{A}_i\emptyset$ has the same multiplicity as $\emptyset$ in $\mathcal{A}_i$.

Capacities: $c(s\mathcal{A}_i) = 1$

$c(\mathcal{A}_i S)$ any positive integer.

$c(St) = \binom{n-l-1}{k-|S|-1}$.